Moufang Loops

Ruth Moufang

- Born Germany, 1905
- Studied at University of Frankfurt
- Received PhD in 1931
- First woman professor in Germany (1957)

Ruth Moufang’s Research

- Used geometry and algebra
- Worked in theoretical physics

Ruth Moufang’s First Degrees of Separation

A Quick Review: Groups

Reminder: A group is a set defined with a binary operation

1. Associative: for a, b, c in G (ab)c = a(bc)
2. Identity: There is an element “e” such that: ae = ea = ea
3. Inverse: for every element a in G, there exists an element a⁻¹ in G such that: aa⁻¹ = e = a⁻¹a

Quasigroups

Definition: A quasigroup is a set Q that has a binary operation *: Q x Q → Q such that
ax = b
ya = b
Where a, b are both in Q and x, y are unique elements of Q.

Note: Quasigroups do not need to be associative
Quasigroups: Examples

Examples:

(Z,·)
(Q\{0},/)

Loops

Definition: A loop is a quasigroup with an identity element, e, such that:
xe = x
ex = x

The identity element is unique and every element of the loop has a left and right inverse.

Loops: Examples

Example:

- Every group is a loop (associative)
- Quaternion group (non-associative)

Moufang Loops!

A Moufang loop is a special case of a loop that satisfies the following properties:

1. \( z(x(zy)) = ((zx)z)y \)
2. \( x(z(yz)) = ((xz)y)z \)
3. \( (zx)(yz) = (z)(xy)z \)
4. \( (zx)yz = z((xy)z) \)

Note: Moufang loops need not be associative

Moufang Loops: Special Properties

- All groups are Moufang loops
- The Moufang loop properties hold in any alternative algebras:
  \( x(xy) = (xx)y \)
  \( y(xx) = (yx)x \)
- Commutative loops only need the single property:
  \( (x'y)z = xyxz \)

Moufang Loops: Two-sided inverse

- Two sided inverse:
  \( x^{-1}(xy) = y = (yx)x^{-1} \)
- Therefore,
  \( (xy)^{-1} = y^{-1}x^{-1} \)
  \( x(yz) = e \) if and only if \( (xy)z = e \)
Moufang Loops: Examples

- All Groups
- Quasigroups that satisfies any one of the Moufang properties
- The Octonions (non-zero and under Octonion multiplication)

Thank you Alissa for a great two weeks 😊 !!