Rouché's Theorem

Preliminary Information – Complex Analysis

- In complex analysis, variables have a real part (x) and an imaginary part (iy), where \( i = \sqrt{-1} \)
  - \( z = x + iy \)

- Functions of complex variables also have a real part and an imaginary part which can be interpreted as *real-valued functions* of the two real variables \( x \) and \( y \).
  - \( f(z) = u(x,y) + iv(x,y) \)
  - Example: \( f(z) = (z^2) = (x + iy)^2 = (x^2 - y^2) + i(2xy) \)

- Addition of complex variables follows the rule:
  - \((a + ib) + (c + id) = (a + c) + i(b + d)\)

- Multiplication of complex variables follows the rule:
  - \((a + ib) \cdot (c + id) = (ac - bd) + i(ad + bc)\)

Preliminary Information – Derivatives

- Taking the derivative of a complex function is *similar* to taking the derivative of a real function, but the limit must have the same value for *any* sequence of complex values for \( z \) that approach the point in question on the complex plane.

\[
f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}
\]
Example: Find the derivative of $f(z) = z^2$.

So,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{(z - z_0)(z + z_0)}{z - z_0} = z + z_0$$

It’s the same as taking the derivative of $f(x) = x^2$.

Critical points of complex functions exist where the derivative is zero.
- Singularities are points $z_0$ in the domain of a function $f(z)$ where $f(z_0)$ fails to be differentiable.
- Poles are singularities at a point $a$ such that $f(z)$ approaches infinity as $z$ approaches $a$.
- When a complex function is differentiable everywhere it is analytic or holomorphic.

Examples of analytic complex functions:
- Polynomials
- Exponentials

Example of a function which is NOT analytic:
- Absolute value function

$$|z| = \sqrt{x^2 - y^2}$$

Not differentiable when $z=0$. 
**Example of a pole:** \( f(z) = \frac{3}{z - 1} \)

**Preliminary Information – Taylor Series**
- The Taylor series we are familiar with holds when the real variable \( x \) is replaced by the complex variable \( z \).
- However, the interval of convergence is now replaced by the idea of the disk of convergence, since the inequality
  \[ |z - z_0| < R \]
  describes the interior of a disk of radius \( R \), centered at the point \( z_0 \).

**Radius of Convergence**

\[ R = \lim_{n \to \infty} \sqrt[n]{|a_n|} \quad \text{where} \quad a_n = \frac{f^{(n)}(z_0)}{n!} \]

- Converges for all \( z \) that satisfy \( |z-z_0| < R \)
- Diverges for all \( z \) such that \( |z-z_0| > R \).
- If \( \lim_{n \to \infty} |a_n| = 0 \), then the series converges for every \( z \). 😊

**Analytic Functions: Redefined.**
- **Theorem:** A function is analytic if and only if it is equal to its Taylor Series in some neighborhood of every point.
Rouche's Theorem

**Rouche's Theorem:** Let two functions $f(z)$ and $g(z)$ be analytic inside and on a simple closed curve $C$, and suppose that $|f(z)| > |g(z)|$ at each point on $C$. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, inside $C$.

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**Geometrically Speaking**

- Picture walking a dog around a tree on a leash with variable length.
- On your walk, you let the leash go long enough so the dog can walk ahead of you. However, provided you keep the leash short enough so the dog cannot reach the tree, you and the dog will still walk around the tree the same number of times!

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**Usage of Rouche's Theorem**

- Rouche's Theorem is used to simplify the problem of finding the zeros in a given analytic function.
- In order to do so, write the function as a sum of two functions, one of which is easier to work with and dominates (grows more quickly than) the other.
- The zeros are then found by looking at only the dominating part of the original function.
Take, for example, the polynomial 

\[ z^7 - 2z^3 + 7 \]

We want to find how many zeros it has in the disk \(|z| < 2\).

Set \( f(z) = z^7 \) and \( g(z) = -2z^3 + 7 \).

Note that \(|g(z)| < |f(z)|\) on \(|z| < 2\).

Since \( f(z) \) has 7 roots in the disk, \( f(z) + g(z) \) also has 7 roots, since it satisfies Rouche’s Hypothesis.

A Little History Lesson

- Eugene Roche, NOT the Eugene we need,
- Fun fact: he was the original Ajax man.

Our Rouché

- Born in southern France,
- Taught at the Charlemagne college and then was professor at the Academy of Arts and Trades in Paris,
- Examiner at the Polytechnic institute,
- Name is known through Rouché’s theorem which he published in the Journal of the École Polytechnique in 1862.

Fundamental Theorem of Algebra

- Rouche’s Theorem can be used to help prove the Fundamental Theorem of Algebra
- The Fundamental Theorem states:
  
  A polynomial 
  
  \[ P(z) = a_nz^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_0 \]
  
  of degree \( n \) always has \( n \) roots.
Proof

Let \( f(z) = a_n z^n \) and \( g(z) = a_{n-1} z^{n-1} + \ldots + a_0 \)

On the circle \( |z| = R \), \( |f(z)| = |a_n| R^n \).

And \( |g(z)| \leq |a_{n-1}| R^{n-1} + \ldots + |a_1| R + a_0 \)

Now, make \( R \) large enough so that
\[
\frac{|a_{n-1}| + \ldots + |a_1| + |a_0|}{|a_n|} < R
\]

Then, \(|f(z)| < |g(z)|\) holds on the boundary of the circle centered at the origin of radius \( R \).

Since \( f(z) \) clearly has \( n \) zeros, we have proved that \( P(z) = f(z) + g(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 \) also has \( n \) zeros using Rouché's Theorem.

Questions?