Rouché's Theorem

Preliminary Information – Complex Analysis

- In complex analysis, variables have a real part (x) and an imaginary part (iy), where $i = \sqrt{-1}$
  - $z = x + iy$
- Functions of complex variables also have a real part and an imaginary part which can be interpreted as *real-valued functions* of the two real variables x and y.
  - $f(z) = u(x,y) + iv(x,y)$
  - Example: $f(z) = (z^2) = (x + iy)^2 = (x^2-y^2) + i(2xy)$

- Addition of complex variables follows the rule:
  - $(a + ib) + (c + id) = (a + c) + i(b + d)$
- Multiplication of complex variables follows the rule:
  - $(a + ib) \bullet (c + id) = (ac – bd) + i(ad + bc)$

Preliminary Information – Derivatives

- Taking the derivative of a complex function is *similar* to taking the derivative of a real function, but the limit must have the same value for *any* sequence of complex values for z that approach the point in question on the complex plane.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Example: Find the derivative of f(z) = z^2.

$$\frac{f(z)-f(z_0)}{z-z_0}=\frac{(z)^2-z_0^{\ 2}}{z-z_0}=\frac{(z-z_0)(z+z_0)}{z-z_0}=z+z_0$$

So,

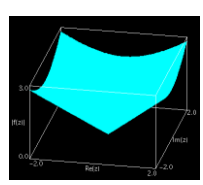
$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\lim_{z\to z_0}(z+z_0)=z_0+z_0=2z_0$$

It's the same as taking the derivative of f(x) = x^2! ☺

- Critical points of complex functions exist where the derivative is zero.
  - Singularities are points z_0 in the domain of a function f(z) where f(z_0) fails to be differentiable.
  - Poles are singularities at a point *a* such that f(z) approaches infinity as z approaches *a.*
- When a complex function is differentiable everywhere it is <u>analytic</u> or <u>holomorphic.</u>

- Examples of analytic complex functions:
  - Polynomials
  - Exponentials

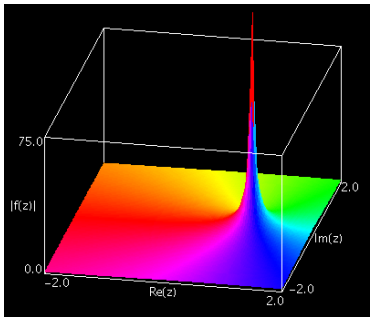
- Example of a function which is NOT analytic:
  - Absolute value function



$$|z|=\sqrt{x^2-y^2}$$

Not differentiable when z=0.

Example of a pole: $f(z) = \frac{3}{z-1}$



Preliminary Information – Taylor Series

- The Taylor series we are familiar with holds when the real variable x is replaced by the complex variable z,
- However, the interval of convergence is now replaced by the idea of the disk of convergence, since the inequality

$$|z - z_0| < R$$

describes the interior of a disk of radius R, centered at the point z_0.

- Radius of Convergence

$$R = \frac{1}{\lim_{n->\infty} \sqrt[n]{|a_n|}} \quad \text{where} \quad a_n = \frac{f^{(n)}(z_0)}{n!}$$

- Converges for all z that satisfy $|z-z_0| < R$
- Diverges for all z such that $|z-z_0| > R$.
- If $\lim_{n->\infty} \sqrt[n]{|a_n|} = 0$, then the series converges for every z. ☺

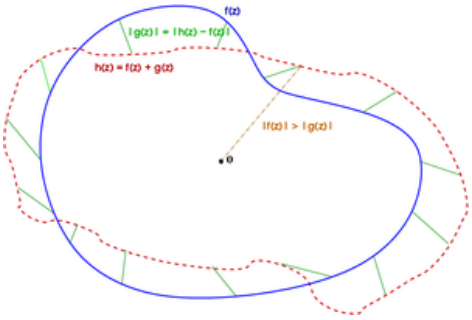
Analytic Functions: Redefined.

- Theorem: A function is analytic if and only if it is equal to its Taylor Series in some neighborhood of every point.

Rouche's Theorem

Rouche's Theorem: Let two functions f(z) and g(z) be analytic inside and on a simple closed curve C, and suppose that |f(z)| > |g(z)| at each point on C. Then f(z) and f(z) + g(z) have the same number of zeros, inside C.

Geometrically Speaking



- Picture walking a dog around a tree on a leash with variable length.
- On your walk, you let the leash go long enough so the dog can walk ahead of you. However, *provided you keep the leash short enough so the dog cannot reach the tree*, you and the dog will still walk around the tree the same number of times!

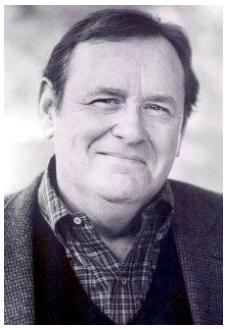
Usage of Rouche's Theorem

- Rouche's Theorem is used to simplify the problem of finding the zeros in a given analytic function.
- In order to do so, write the function as a sum of two functions, one of which is easier to work with and dominates (grows more quickly than) the other.
- The zeros are then found by looking at only the dominating part of the original function.

- Take, for example, the polynomial $z^7 - 2z^3 + 7$
- We want to find how many zeros it has in the disk $|z|<2$.
- Set $f(z) = z^7$ and $g(z) = -2z^3 + 7$.
- Note that $|g(z)|<|f(z)|$ on $|z|<2$.
- Since f(z) has 7 roots in the disk, f(z) + g(z) also has 7 roots, since it satisfies Rouche's Hypothesis.

A Little History Lesson

- Eugene Roche,
- NOT the Eugene we need,
- Fun fact: he was the original Ajax man.



Our Rouché

- Born in southern France,
- Taught at the Charlemagne college and then was professor at the Academy of Arts and Trades in Paris,
- Examiner at the Polytechnic institute,
- Name is known through Rouché's theorem which he published in the *Journal of the École Polytechnique* in 1862.

Fundamental Theorem of Algebra

- Rouche's Theorem can be used to help prove the **Fundamental Theorem of Algebra**
- The Fundamental Theorem states:

A polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_0$$

of degree n always has n roots.

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_0$$

Proof

Let $f(z) = a_n z^n$ and $g(z) = a_{n-1} z^{n-1} + \ldots + a_0$

On the circle $|z| = R$, $|f(z)| = |a_n| R^n$,

And $|g(z)| \le |a_{n-1}| R^{n-1} + \ldots + |a_1| R + a_0$

Now, make R large enough so that

$$\frac{|a_{n-1}| + \ldots + |a_1| + |a_0|}{|a_n|} < R$$

Then, $|f(z)|<|g(z)|$ holds on the boundary of the circle centered at the origin of radius R. Since f(z) clearly has *n* zeros, we have proved that

$$P(z) = f(z) + g(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$$

also has n zeros using Rouché's Theorem.

Questions?