3

What Comes Next?

We hope you like the kind of problem where someone gives you an intriguing number sequence and asks you what comes next. In this chapter we'll give you several ways to find out. Most of these involve building some kind of pattern from your numbers. Pascal's triangle is one very well-known pattern.

If you're like us, you'll also enjoy playing with number sequences for their own sake, so we'll also show you some number games that often use patterns to magically transform one sequence into another. One of the nicest of these is described next.

Moessner's Magic

Alfred Moessner discovered that some of our favorite sequences can be found in a surprising new way. Start with the counting numbers and circle every second number; then form the cumulative totals of the uncircled numbers, and you'll see the squares:

1	(2)	3	(4)	5	(6)	7	(8)	9	(10)	11	(12)	13	(14)	15	(16)
(1)		(4)		(9)		(16)		(25)		(36)		(49)		(64)	

If instead you circle every third number, total what's left, circling the last number in each block, and total the uncircled (hex) numbers, you'll see the cubes:

1	2	(3)	4	5	(6)	7	8	(9)	10	11	(12)	13	14	(15)
1	(3)		7	(12)		19	(27)		37	(48)		61	(75)	
(1)			(8)			(27)			(64)			(125)		

Circling every fourth number:

1	2	3	(4)	5	6	7	(8)	9	10	11	(12)	13	14	15	(16)
1	3	(6)		11	17	(24)		33	43	(54)		67	81	(96)	
1	(4)			15	(32)			65	(108)			175	(256)		
(1)				(16)				(81)				(256)			

leads similarly to the fourth powers, and so on.

So circling the numbers

$$n + n \quad n + n + n \quad n + n + n + n \ldots$$

has led to the numbers

$$n \times n \quad n \times n \times n \quad n \times n \times n \times n \ldots.$$

If we circle each triangular number, $1 + 2 + 3 + \cdots + n$:

(1)	2	(3)	4	5	(6)	7	8	9	(10)	11	12	13	14	(15)
	(2)		6	(11)		18	26	(35)		46	58	71	(85)	
			(6)			24	(50)			96	154	(225)		
						(24)				120	(274)			
										(120)				

we get the **factorial numbers**, $1 \times 2 \times 3 \times \cdots \times n$, which we'll talk about soon.

What if we circle the squares?

(1)	2	3	(4)	5	6	7	8	(9)	10	11	12	13	14	15	(16)
	2	(5)		10	16	23	(31)		41	52	64	77	91	(106)	
	(2)			12	28	(51)			92	144	208	285	(376)		
				12	(40)				132	276	484	(769)			
				(12)					144	420	(904)				
									144	(564)					
									(144)						

If these numbers mystify you, notice that the squares are

$$1$$
$$1 + 2 + 1$$
$$1 + 2 + 3 + 2 + 1$$
$$1 + 2 + 3 + 4 + 3 + 2 + 1$$
$$\dots$$

and that the final circled numbers are

$$1$$
$$1 \times 2 \times 1$$
$$1 \times 2 \times 3 \times 2 \times 1$$
$$1 \times 2 \times 3 \times 4 \times 3 \times 2 \times 1$$

The general rule is that if you start by circling

$$1a, \quad 2a + 1b, \quad 3a + 2b + 1c, \quad 4a + 3b + 2c + 1d \dots,$$

then the final circled numbers are

$$1^a, \quad 2^a \times 1^b, \quad 3^a \times 2^b \times 1^c, \quad 4^a \times 3^b \times 2^c \times 1^d \dots.$$

FACTORIAL NUMBERS

How many "words" can we make from the letters A, E, T, each used just once?

AET, ATE, EAT, ETA, TAE, TEA

The first letter can be any one of the three, the second can be either one of the two remaining, and the third is then the one left over,

$$3 \times 2 \times 1 = 6 \text{ words.}$$

If you have n different letters, they can be arranged in

$$n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1 \text{ ways.}$$

This number is called **factorial** n, or n **factorial**. It often used to be written $\underline{\lfloor n}$, but today is usually written $n!$.

Of course, there's just one way to arrange no objects, so $0! = 1$. In general, $n!$ is the product of the numbers from 1 to n, the empty product being 1 (Figure 3.1).

	=	1	= 0!
1	=	1	= 1!
1 × 2	=	2	= 2!
1 × 2 × 3	=	6	= 3!
1 × 2 × 3 × 4	=	24	= 4!
1 × 2 × 3 × 4 × 5	=	120	= 5!
1 × 2 × 3 × 4 × 5 × 6	=	720	= 6!
1 × 2 × 3 × 4 × 5 × 6 × 7	=	5040	= 7!

FIGURE 3.1 *The factorial numbers.*

We just saw how we can get the factorial numbers from Moessner's magic, and in fact we already met them in Chapter 2 when we piled up triangular pyramids in more and more dimensions.

ARRANGEMENT NUMBERS

Factorial n is the number of **arrangements**, or **orders**, or **permutations** of n things in a row. How many arrangements are there of r objects, chosen from n different things? The first can be any one of the n, the second can be any one of the remaining $n-1$, the third any one of remaining $n-2$, and so on, the rth being any one of $n - r + 1$. The total number of different arrangements is

$$n \times (n-1) \times (n-2) \times \cdots \times (n-r+1),$$

the product of all the numbers from 1 to n, except for those from 1 to $n - r$, so we can express this concisely using the factorial numbers:

> The number of **arrangements** of r things out of n is
> $$\frac{n!}{(n-r)!}$$

CHOICE NUMBERS

If we're only concerned with the number of **choices**, or **combinations**, of the r things chosen from the n, then we don't distinguish between the factorial r different ways in which we could have arranged them in a row. So to get the **choice numbers**, $\binom{n}{r}$, we divide the arrangement numbers by r!

The number of *choices* of r things from n is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

In this formula, you can swap r for $n - r$ without altering the value. The number of ways of choosing 5 things out of 8 is the same as the number of ways of choosing the 3 you want to leave out:

$$\binom{8}{5} = \binom{8}{3}$$

and generally,

$$\binom{n}{r} = \binom{n}{n-r}$$

This is the left-right symmetry of Pascal's triangle, see Figures 3.2 and 3.3

Suppose a class of 28 students wants to choose a soccer team of 11 players. In how many ways can they do it? We now know that this is

$$\binom{28}{11} = \frac{28 \times 27 \times 26 \times 25 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19 \times 18}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11}$$

$$= \frac{28!}{11!17!} = 2^2 \times 3^3 \times 5 \times 7 \times 13 \times 19 \times 23 = 21474180.$$

Now suppose you're in the class and want to know if you're on the team. In how many ways could you be included? If you're on, the other 10 must be chosen from the other 27:

$$\binom{27}{10} = 8436285 \text{ ways.}$$

In how many ways are you *not* included? All 11 have to be chosen from the other 27:

$$\binom{27}{11} = 13037895 \text{ ways.}$$

So $\binom{28}{11}$ is the sum of these two numbers, and generally, since you are either on the team of r or not,

$$\boxed{\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}}$$

PASCAL'S TRIANGLE

This is a very simple way of generating the choice numbers. Start from $\binom{0}{0} = 1$ on row 0, and $\binom{1}{0} = 1$ and $\binom{1}{1} = 1$ on row 1, and calculate successive rows by putting $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ at each end and forming each other number as the sum of the two in the row immediately above (Figure 3.2).

The first few choice numbers are shown in Figure 3.2. The array in Figure 3.3 is usually known as **Pascal's triangle**, because it was intensively studied by Blaise Pascal (1623–1662), the famous French philosopher and mathematician. It had already been described much earlier by Chinese mathematicians and by Omar Khayyám, who died in 1123.

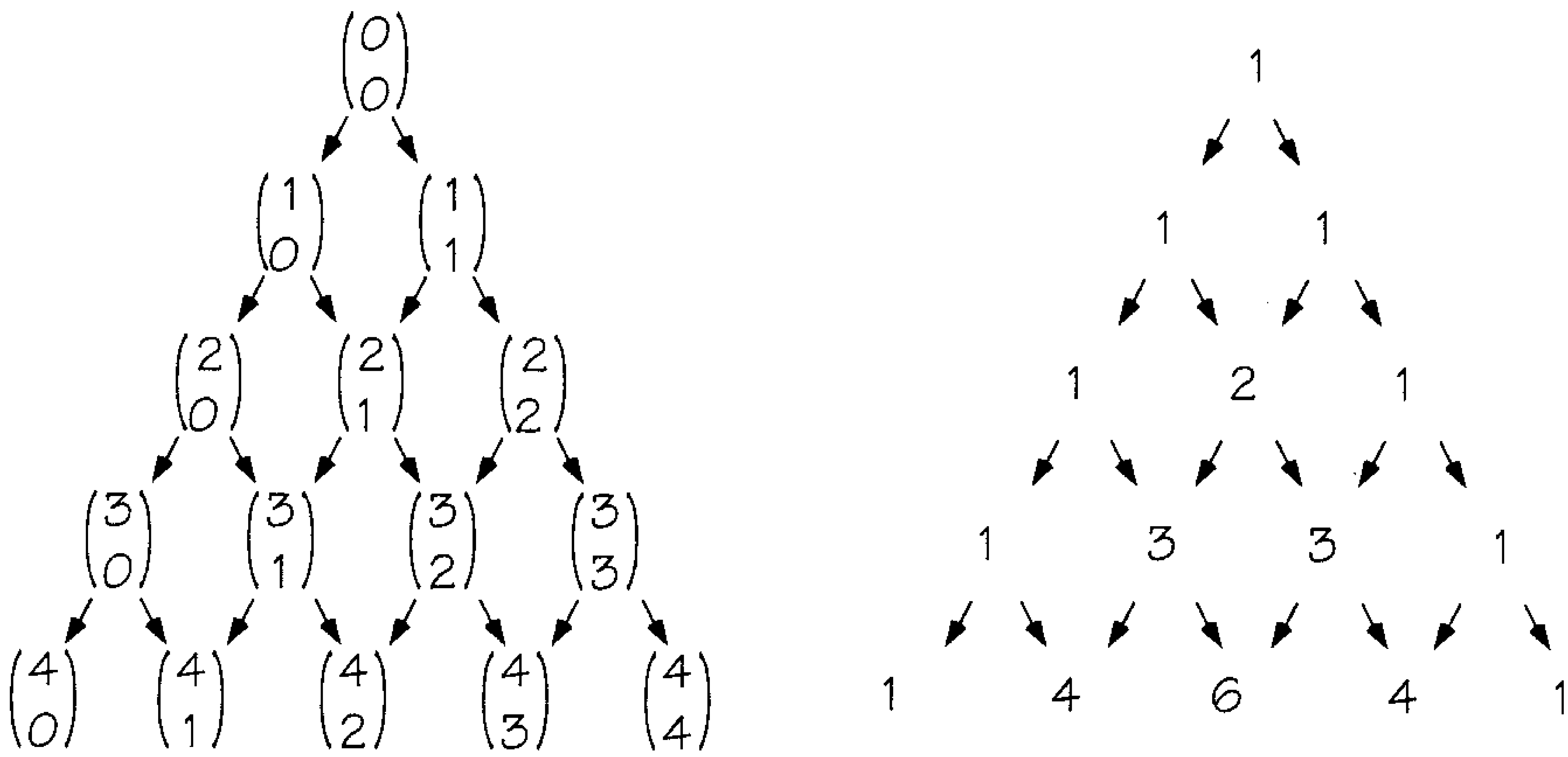

FIGURE 3.2 *Generating the choice numbers.*

Of course, we've seen some of these numbers before, in Chapter 2, when we piled up triangular pyramids in more and more dimensions. The numbers at the beginning of each row are just ones,

1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...

The second numbers in each row are the **counting numbers**,

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1

FIGURE 3.3 *Pascal's numbers: the choice numbers, or binomial coefficients.*

The third numbers are the **triangular numbers**,

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, \ldots.$$

The fourth numbers are the **tetrahedral numbers**,

$$1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286, \ldots.$$

The fifth ones are the **pentatope numbers**,

$$1, 5, 15, 35, 70, 126, 210, 330, 495, 715, 1001, \ldots,$$

and so on. The numbers in each diagonal are the cumulative sums of those in the previous diagonal.

CHOICE NUMBERS WITH REPETITIONS

In how many ways can you choose five things from n, if repetitions are allowed? In other words, how many essentially different kinds of "poker hands" are there, if we ignore flushes and straights and are playing with a double deck, so that you can have five of a kind?

"Poker hand"	13 in a suit	n cards in a suit
all different	$\binom{13}{5} = 1287$	$\binom{n}{5}$
one pair	$13 \times \binom{12}{3} = 2860$	$n \times \binom{n-1}{3}$
two pairs	$\binom{13}{2} \times 11 = 858$	$\binom{n}{2} \times (n-2)$
three of a kind	$13 \times \binom{12}{2} = 858$	$n \times \binom{n-1}{2}$
full house (3 & 2)	$13 \times 12 = 156$	$n \times (n-1)$
four of a kind	$13 \times 12 = 156$	$n \times (n-1)$
five of a kind	$13 = 13$	n
Total	$6188 = \binom{17}{5}$	$\binom{n+4}{5}$

Surely such a simple answer can be found more simply? In fact, the hands correspond to the number of 5-card hands chosen from a *Sweet Seventeen* deck of 17 distinguishable cards: A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, and 2 and four distinguishable jokers: j_1, j_2, j_3, and j_4.

If you are dealt a Sweet Seventeen hand (Figure 3.4(a)), sort it in the usual way, high on the left, low on the right, but with any jokers in the positions corresponding to their labels (Figure 3.4(b)). To convert it into a poker hand, replace any jokers by copies of the first genuine card that follows them: Figure 3.4(c) shows the resulting full house, nines on twos.

To see why the correspondence is exact, convert your sorted poker hands (Figure 3.5(a)) into a Sweet Seventeen hand by replacing all duplicates of cards farther to the right by jokers, labeled by their position counting from the left (Figure 3.5(b)).

In general, to find the number of choices of r things from n different ones, but with repetitions allowed, imagine that you are playing Sweet Seventeen, but instead of a deck of 13 + 4 jokers, you have a deck of $n + (r - 1)$ jokers, and the answer is

$$\binom{n + r - 1}{r}.$$

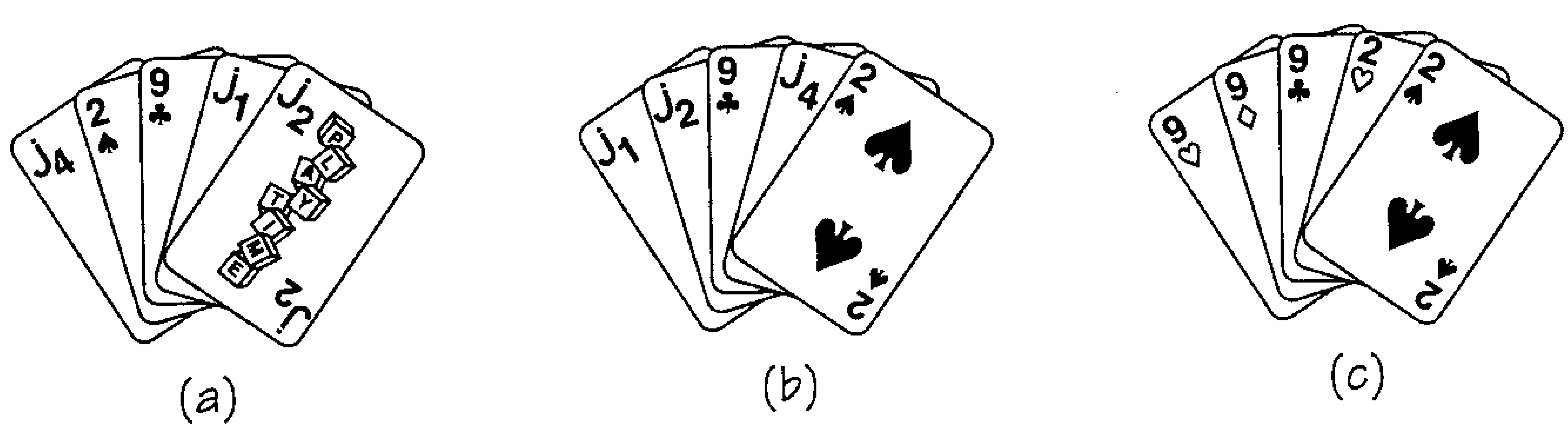

FIGURE 3.4 *A "Sweet Seventeen" hand becomes a poker hand.*

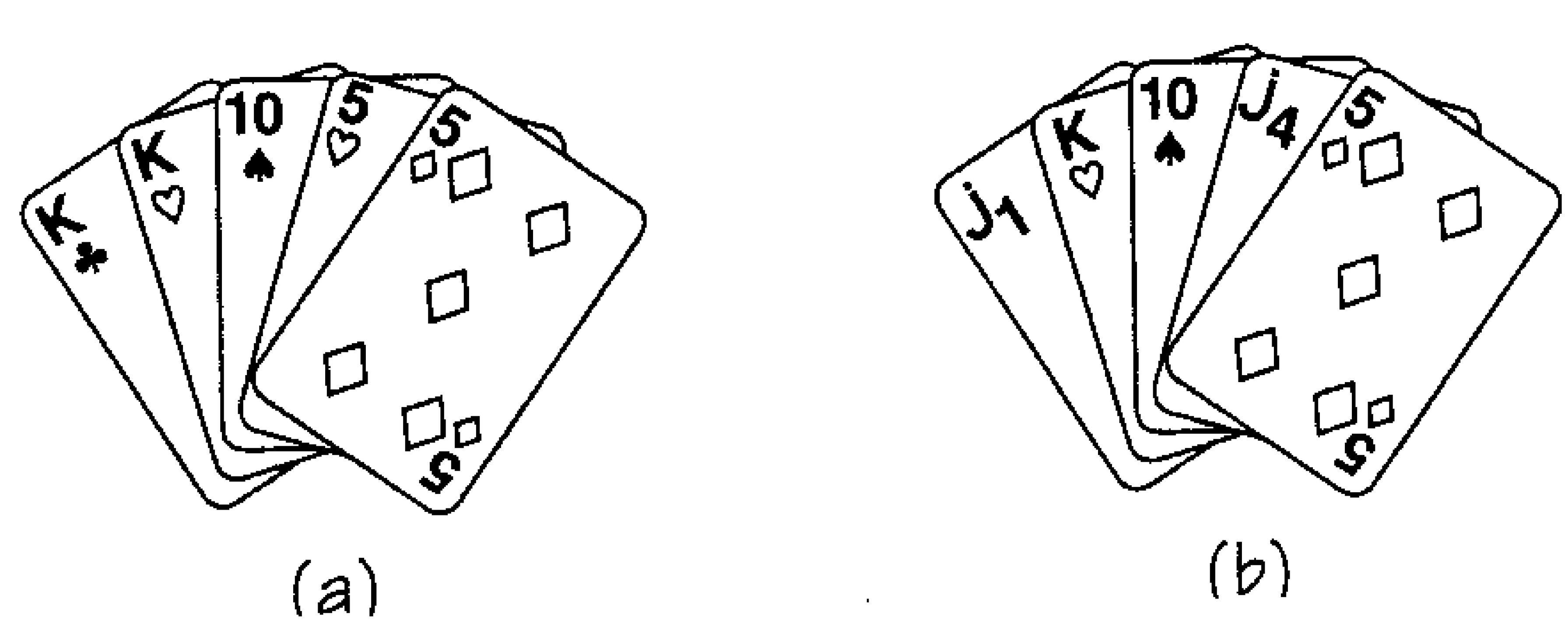

FIGURE 3.5 *A poker hand becomes a Sweet Seventeen hand.*