Population Modeling with Ordinary Differential Equations

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Abstract

Population modeling is a common application of ordinary differential equations and can be studied even the linear case. We will investigate some cases of differential equations beyond the separable case and then expand to some basic systems of ordinary differential equations. The phase line and phase plane will be used to assist in plotting the solutions of these systems and consequently, to aide understanding the behavior of a hypothetical environment over time.

Contents

1 Introduction

1.1 Simple Ordinary Differential Equations

The simplest type of differential equation is the standard case you find in calculus

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- **Completely non–autonomous** differential equations. These are nothing more than some of those MATH–032 integrals.

  Example:

  $$\frac{dx}{dt} = t^2 + t$$

  Solution:

  $$x(t) = \int \left(t^2 + t\right) dt = \frac{t^3}{3} + \frac{t^2}{2} + C.$$

- **Separable** differential equations are types that you've probably encountered before and are not too hard to work out.

  Example:

  $$\frac{dx}{dt} = (x + 1)\, t$$

  Solution:

  Separate variables and integrate both sides with respect to the given variable.

  $$\int \frac{dx}{x + 1} = \int t\, dt$$

  Then we can integrate these equations to obtain a general solution

  $$\ln \left| x + 1 \right| = \frac{t^2}{2} + C$$

1.2 Problems with Analytic Techniques

There are many different examples that arise in differential equations which raise problems with these analytic techniques. Here are just a couple.

Example: The following differential equation is separable. [1]

$$\frac{dx}{dt} = \frac{x}{x + 1}$$

Solution: We can separate and integrate easily as follows.

$$\int \left(\frac{1}{x} + x \right) dx = \int dt$$

Problem: There is no way to algebraically solve the equation

$$\ln|x| + \frac{x^2}{2} = t + C.$$

Example: This is a perfectly separable differential equation. [1]

$$\frac{dx}{dt} = \sec(x^2)$$

Problem: It's a pity the left hand integration is perhaps impossible.

$$\int \cos(x^2)\, dx = \int dt$$

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2 Definitions and Terminology

2.1 Ordinary Differential Equation

An **ordinary differential equation** [2] relates the ordinary derivatives of an unknown function and possibly the function $x(t)$ itself. It has the general form

$$G\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \frac{d^3x}{dt^3}, \ldots\right) = 0.$$

where the function G determines which derivatives are involved in the equations and the extent to which each is involved. Familiar differential equations, such as the following, take this form.

Examples:

$$t^2 + t - \frac{dx}{dt} = 0$$

$$x + \frac{dx}{dt} = 0$$

$$x^2 + x + \frac{d^4x}{dt^4} = 0$$

Newton's Law: One the most common differential equations used in physical application is Newton's

$$F = ma.$$

Acceleration is the second derivative of a displacement function $x(t)$ so, we have the differential equation

$$F - m\frac{d^2x}{dt^2} = 0.$$

2.2 Initial Value Problems

- An **initial value problem** involves both a differential equation and a prescribed value for the unknown function $x(t)$ at some initial time t_0. Express such a problem as

$$F\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \frac{d^3x}{dt^3}, \ldots\right) = 0 \qquad \text{with} \qquad x(t_0) = x_0$$

- A solution to an initial value problem must satisfy both the differential equation as well as the initial value prescribed at the particular initial time t_0.

2.3 Equilibrium Points

An **equilibrium point** x_* is one where a solution of the differential equation remains fixed at x_* for all time. In other words $x(t) = x_*$ for all $t \in \mathbf{R}$. Equivalently, equilibrium points are determined by locating the points where

$$\frac{dx}{dt} = 0.$$

For a system of differential equations, we will have something like:

$$\frac{dx}{dt} = f(x, y) \qquad \text{and} \qquad \frac{dy}{dt} = g(x, y).$$

In this instance, **equilibrium points** are all such points (x_*, y_*) which satisfy both

$$f(x, y) = 0 \qquad \text{and} \qquad g(x, y) = 0.$$

3 Single Species Population Models

3.1 Exponential Growth

We just need one population variable in this case. The simplest (yet–incomplete model) is modeled by the rate of growth being equal to the size of the population.

Exponential Growth Model: A differential equation of the separable class.

$$\frac{dP}{dt} = kP \qquad \text{with} \qquad P(0) = P_0$$

We can integrate this one to obtain

$$\int \frac{dP}{kP} = \int dt \qquad \Longrightarrow \qquad P(t) = Ae^{kt}$$

where A derives from the constant of integration and is calculated using the initial condition. This solution may be easier to see on a phase line.

3.2 Logistic Model Growth

Exponential growth is not quite accurate since the environmental support system for a given species is likely *not* infinite.

Logistic Model:

$$\frac{dP}{dt} = kP(M - P)$$

where M is some maximum population or what environmentalists might call the "carrying capacity".

We can better capture the behavior of a population model on a phase line and derivative field.

4 Linear Systems of Ordinary Differential Equations

For linear systems, we consider the effects of two unknown functions on each other. A two by two system of linear ordinary differential equations has the form

$$\begin{cases} \frac{dx}{dt} = ax + by \\ \\ \frac{dy}{dt} = cx + dy \end{cases} \qquad \text{or} \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

If we set $\mathbf{Y}$ to be the unknown vector and let $\mathbf{A}$ be the coefficient matrix, then we have a matrix equation

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}. \tag{1}$$

4.1 Source and Sink Equilibria

Theorem: [1] If $\mathbf{A}$ has distinct real eigenvalues λ_1 and λ_2 with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then the general solution of (1) is

$$\mathbf{Y}(t) = k_1 \exp(\lambda_1 t)\,\mathbf{v}_1 + k_2 \exp(\lambda_2 t)\,\mathbf{v}_2.$$

Example: An unstable system of differential equations

$$\begin{cases} \frac{dx}{dt} = x \\ \frac{dy}{dt} = 2y \end{cases} \qquad \text{or} \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

The coefficient matrix $\mathbf{A}$ has $\lambda_+ = 2$ and $\lambda_- = 1$ with eigenvectors $\mathbf{v}_+ = (0,1)$ and $\mathbf{v}_- = (1,0)$. Then the general solution is

$$\mathbf{Y}(t) = k_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

4.2 Periodic Equilibria

Theorem: [1] If the eigenvalues of the coefficient matrix $\mathbf{A}$ are complex (having the form $\lambda = \alpha \pm i\beta$), then the solution of (1) is

$$\mathbf{Y}(t) = e^{\alpha t}\Big(\cos(\beta t) \ + \ i\sin(\beta t)\Big)\mathbf{v}$$

where $\mathbf{v}$ is an eigenvector of $\mathbf{A}$.

Example: Periodic solutions.

$$\begin{cases} \frac{dx}{dt} & = \ -y \\ \\ \frac{dy}{dt} & = \ x \end{cases} \qquad \text{or} \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

We get $\text{Char}(\mathbf{A}) \ = \ \lambda^2 \ + \ 1 \ = \ 0$. Hence $\lambda \ = \ \pm i$.
Only one of the eigenvectors is necessary in this case. Take $\mathbf{v} \ = \ \begin{bmatrix} i \\ 1 \end{bmatrix}$, corresponding to $\lambda \ = \ i$ and coming from

$$\mathbf{Av} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \ = \ \begin{bmatrix} iv_1 \\ iv_2 \end{bmatrix} = \lambda\mathbf{v}$$

So, the solution works out to be

$$\mathbf{Y}(t) = \Big(\cos(t) \ + \ i\sin(t)\Big)\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin(t) + i\cos(t) \\ \cos(t) + i\sin(t) \end{bmatrix}$$

Then we have

$$\mathbf{Y}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + i\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \mathbf{Y}_{\text{Re}}(t) + i\mathbf{Y}_{\text{Im}}(t)$$

The real and imaginary parts are each independent solutions to the system of differential equations.

4.3 Trace – Determinant Plane

One can essentially learn about the behavior of a linear system of the form

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

simply by knowing the eigenvalues of the coefficient matrix. These are encapsulated in the characteristic equation

$$\text{Char}(\mathbf{A}) \;=\; \lambda^2 \;-\; \text{Tr}(\mathbf{A})\lambda \;+\; \text{Det}(\mathbf{A}) \;=\; 0.$$

Let T denote the trace and D the determinant of the matrix so that we may write (more compactly) the discriminant of this quadratic:

$$T^2 - 4D.$$

When this quantity is larger than 0, there exist two real roots (hence there are two real eigenvalues for the system).

When this quantity is less than 0, there exist two complex roots (and hence there are two complex eigenvalues for the system).

To better assess how differing values of D and T affect the eigenvalues, we plot the curve $D = \frac{T^2}{4}$ in the plane and isolate regions of varying eigenvalue characteristics.

5 Non–linear Systems of Ordinary Differential Equations

5.1 Linearization of a System

We will first determine some global properties of the system and then linearize to approximate some more local behavior.

A **nullcline** is a line in the phase plane where either the rate of change of x vanishes (x–nullcline) or where the rate of change of y vanishes (y–nullclines). Nullclines help determine global behavior.

Example: Consider the system of differential equations given by

$$\frac{dx}{dt} = x\,(2 - x) - xy$$

$$\frac{dy}{dt} = y\,(3 - y) - 2xy$$

The x–nullclines are lines that satisfy the equation

$$\frac{dx}{dt} = x\,(2 - x) - xy = x\,(2 - x - y) = 0.$$

So, we find x–nullclines of $x = 0$ and $y = 2 - x$.

The y–nullclines need to satisfy the equation

$$\frac{dy}{dt} = y\,(3 - y) - 2xy = y\,(3 - y - 2x) = 0.$$

So, the y–nullclines are $y = 0$ and $y = 3 - 2x$.

5.2 Population Models with Non–linear Systems

Just like many phenomena in calculus, we can discover a sufficient amount of information from these systems by linearizing them.

We can linearize the system at each equilibrium point to learn how the solution curves behave locally for each of those points.

Proposition: The solutions to the linearized system near an equilibrium point are a close approximation to the solutions of the actual system provided that the linearized system is neither a center nor a system with a zero eigenvalue.

To linearize a system of differential equations given by

$$\frac{dx}{dt} = f(x,y) \qquad \text{and} \qquad \frac{dy}{dt} = g(x,y),$$

at an equilibrium point (x_0, y_0), we use the system

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix}_{(x_0,y_0)} \mathbf{Y}.$$

Example: Consider the system of differential equations given in (5.1)

The linearized system at the equilibrium point (0,0) is

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{Y}.$$

The eigenvalues are $\lambda_+ = 3$ and $\lambda_- = 2$, so the system behaves like a source in the viscinity of (0,0).

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6 Hypothetical Environment Project

- For the following differential equation, fill in some appropriate constants to define the rate of growth of your *rabbit* population. Recall that in the equation

$$\frac{dR}{dt} = aR - bRF$$

  a represents the growth rate of your rabbit population and b represents the effect of the foxes preying on your rabbits.

$$a = \_\_\_\_\_\_\_\_ > 0 \qquad\qquad b = \_\_\_\_\_\_\_\_ > 0$$

- Find a partner in the room who has a differential equation for a fox population. Combine your models to form a system of ordinary differential equations representing a predator–prey system.

$$\begin{cases} \frac{dR}{dt} = aR - bRF \\ \\ \frac{dF}{dt} = -cF + dRF \end{cases}$$

- Use R and F nullclines and linearization to determine the behavior of your model and whether your populations survive harmoniously or not. Do the initial conditions have a drastic impact on the outcome of your environment?

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