Generating Random Wishart Matrices with Fractional Degrees of Freedom in OX

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Abstract

Several recently proposed Multivariate Stochastic Volatility (MSV) models suggest using Wishart processes to capture the dynamic correlation structure of asset returns. When estimating these models with Markov chain Monte Carlo (MCMC) methods, the degrees of freedom (df) in the Wishart sampling may be fractional (non-integer), e.g. Asai and McAleer (2009). In such cases, if we simply use integer df for the MCMC updates, the model estimation would be seriously misled. This motivates our need for the generation of Wishart random matrices having fractional df. In this paper, based on Bartlett's decomposition, we develop the OX package "WishPack" for generating Wishart/inverse-Wishart random matrices with fractional df. To make the package more complete, the density functions for the Wishart and inverse Wishart distributions are also provided. An additional feature of this package is that it takes into account the singular Wishart matrices and distributions, since they have been well defined and are useful in practical problems.

Keywords: Bartlett's decomposition; Singular Wishart distribution; Wishart matrix.

1 Introduction

Asset correlations are well known to be unstable over time. Therefore, in financial modelling, capturing the dynamic covariance and correlation structure has become a central issue. In the literature of Multivariate Stochastic Volatility (MSV), several recently proposed models suggest using Wishart processes to characterize the evolution of covariance matrices. For example, Philipov and Glickman (2006) and Asai and McAleer (2009; henceforth A&M) adopt an inverse Wishart specification for the covariance matrices, so that statistical inference and model estimation can be effected through Bayesian Markov chain Monte Carlo (MCMC) methods. Following A&M, this class of models is termed the "Wishart Inverse Covariance" (WIC).

Since the WIC model for time series data is built upon latent variables and its estimation relies on MCMC approaches, the computation is expensive. To deal with this type of model, OX would be an ideal choice, as it is fast and it has extensive and well developed packages for Bayesian inference. A problem arises here: in A&M's model settings, the degrees of freedom (df) parameter in the Wishart distribution is not restricted to be integer-valued; it could be fractional (non-integer). However, the generation of Wishart matrices with fractional df is not available in OX. The existing OX function "ranwishart" will simply take the floor of any fractional df as the input argument and thus the matrix that has been sampled is in fact from a Wishart distribution with integer df, which is not correct. In such cases where the integer df is used, our simulation shows that the estimation will be seriously misled. To fix the problem in OX, we develop the Wishart package, "WishPack", in which fractional df is considered.

Moreover, when the df is less than the dimension of the scale matrix parameter, the Wishart distribution is known to be singular. In this situation, the Wishart matrix is still well defined, and the probability density function with respect to Lebesgue measure on an appropriate space is explicitly given by Srivastava (2003). Practical uses of the singular Wishart matrices and distributions are revealed in Uhlig (1997), Kubokawa and Srivastava (2008), and Kunno (2009), among others. In response to those needs, Wishpack also takes into account the singular Wishart cases. This special feature concerning modern Wishart research has not yet been seen in other statistical packages. Finally, as a parallel product, Wishpack also provides an inverse Wishart generator with fractional df and the density function of the inverse Wishart distribution.

The rest of the paper is organized as follows. In Section 2 we motivate the generation of Wishart matrices with fractional df. The generation of the singular Wishart matrix is also discussed. Section 3 illustrates the algorithm for generating Wishart random matrices with fractional df by applying Bartlett's decomposition. A short discussion of another popular method, namely the Odell-Feiveson's (1966) algorithm, is given as well. Some concluding remarks are summarized in Section 4.

2 Motivation

Let $W(\cdot,\cdot)$ be the Wishart distribution. A&M's dynamic correlation MSV model is given by

$$\boldsymbol{Q}_t^{-1}|k, \boldsymbol{S}_{t-1} \sim W(k, \boldsymbol{S}_{t-1}),$$

$$\boldsymbol{S}_{t-1} = \frac{1}{k}\boldsymbol{Q}_{t-1}^{-d/2}\boldsymbol{A}\boldsymbol{Q}_{t-1}^{-d/2},$$

where $\boldsymbol{Q}_t$ is a positive definite matrix, by standardizing which we obtain the correlation matrix of interest, and k and $\boldsymbol{S}_{t-1}$ are the df and the time-varying scale matrix of the Wishart distribution, respectively. The matrix $\boldsymbol{A}$ is the intertemporal sensitivity parameter which is symmetric and positive definite, and d is the persistence parameter, which accounts for the memory of the matrix process $\{\boldsymbol{Q}_t\}$. The matrix $\boldsymbol{Q}_{t-1}^{-d/2}$ is defined by a singular value decomposition.

In the MCMC estimation procedure, the full conditionals of $\boldsymbol{Q}_t^{-1}$ and $\boldsymbol{A}$ are proportional to the Wishart densities with fractional df $\hat{k}$ and $\hat{\gamma}$, respectively. To demonstrate why "fractional" does matter, we begin with the simulation example 3.3 in A&M, in which the sample size $T = 500$ and the true parameters are

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1.10 & -0.33 \\ -0.33 & 1.10 \end{pmatrix}, \ d = 0.8, \ k = 10.$$

Let *df* denote the set $\{\hat{k}, \hat{\gamma}\}$. Table 1 shows the results of the MCMC estimation when *df* is taken to be integer-valued and to be fractional, respectively. The 95% intervals are obtained using the (2.5th, 97.5th) percentiles. Note that for the integer *df* case, the OX "ranwishart" function is used, while for the fractional *df*, we apply WishPack. The MCMC simulation is conducted with only 1000 iterations since a systematic collapse will occur very soon in the integer *df* case. The first 400 draws are discarded and the remaining 600 are kept. Here we have to emphasize that the purpose of this simulation study is to illustrate the problem, not to estimate the model, hence the numerical standard errors are not provided and the convergence diagnostic is not implemented. From Table 1, we can see that the estimates for $\boldsymbol{A}$ and k are totally misleading for the integer-valued *df*. As a result, a systematic failure will happen in this case due to the repeatedly incorrect updates. On the other hand, if a fractional *df* is adopted, then the MCMC estimation procedure will provide a fairly good result.

Figure 1 shows the trace plots of the entire chains of k and $|\boldsymbol{A}|$ under different types of *df*. Here $|\cdot|$ denotes the determinant operator. It is readily seen that, in the integer *df* case, $|\boldsymbol{A}|$ goes down to nearly zero within the first 300 iterations and never moves back. This finally leads us to a systematic error. On the contrary, for the fractional *df*, after the burn-in period, k and $|\boldsymbol{A}|$ both stay stable around their true levels. This simple simulation clearly demonstrates the need for the Wishart generator with fractional df in real applications. As a matter of fact, since the set of natural numbers is a subset of the positive fractional numbers, we can always apply fractional df in the Wishart sampling.

Table 1. MCMC results using the integer and the fractional df parameters, respectively.

	Integer *df*		Fractional *df*	
Parameters	Mean	95% Interval	Mean	95% Interval
a_{11}	0.100	(0.072, 0.148)	1.010	(0.940, 1.092)
a_{12}	-0.027	(-0.053, -0.014)	-0.374	(-0.581, -0.255)
a_{22}	0.103	(0.069, 0.155)	1.161	(1.090, 1.239)
d	0.797	(0.728, 0.858)	0.763	(0.638, 0.831)
k	4.955	(4.179, 5.842)	7.574	(5.667, 9.533)

3 Generating Wishart Random Matrices with Fractional df

3.1 Non-singular Wishart Matrices

Let $W(k, \boldsymbol{S})$ be the Wishart distribution with the fractional df k and the $p \times p$ scale matrix $\boldsymbol{S}$. Also let $\Gamma(\,\cdot\,, \frac{1}{2})$ be the gamma distribution with rate $\frac{1}{2}$. If $k \geq p$, then we can generate random matrices from $W(k, \boldsymbol{S})$ with the following algorithm:

(1) Generate the random matrix $\boldsymbol{B}$ by

i. $\boldsymbol{B}_{ii} \sim \Gamma\left(k - i + 1, \frac{1}{2}\right)$, for $1 \leq i \leq p$;

ii. $\boldsymbol{B}_{ij} \sim N(0, 1)$, $1 \leq j < i \leq p$, and $\boldsymbol{B}_{ij} = 0$, o.w.

(2) Take Cholesky decomposition for the scale matrix $\boldsymbol{S} = \boldsymbol{A}\boldsymbol{A}^T$.

(3) $\boldsymbol{W} \equiv \boldsymbol{A}\boldsymbol{B}\boldsymbol{B}^T\boldsymbol{A}^T = (\boldsymbol{A}\boldsymbol{B})(\boldsymbol{A}\boldsymbol{B})^T \sim W(k, \boldsymbol{S})$.

This algorithm is based on Bartlett's decomposition (Anderson, 2003). To verify that $\boldsymbol{W}$ is actually drawn from $W(k, \boldsymbol{S})$, we conduct a simulation study and check if the results satisfy the following property (Gupta and Nagar, 2000, Corollary 3.3.11.1):

$$\text{If } \boldsymbol{W} \sim W(k, \boldsymbol{S}), \text{ then } \frac{\boldsymbol{c}^T\boldsymbol{W}\boldsymbol{c}}{\boldsymbol{c}^T\boldsymbol{S}\boldsymbol{c}} \sim \chi^2(k), \ \forall \boldsymbol{c}(p \times 1) \neq \boldsymbol{0}, \tag{1}$$

where $\chi^2(k)$ is the chi-square distribution with df $= k$. For the verification, four vectors are chosen, which are $\boldsymbol{c}_1 = (1, 0, 0)^T$, $\boldsymbol{c}_2 = (0, 1, 0)^T$, $\boldsymbol{c}_3 = (0, 0, 1)^T$, and $\boldsymbol{c}_4 = (1, 1, 1)^T$. Next, define $\chi_i^2 \equiv \left(\boldsymbol{c}_i^T\boldsymbol{W}\boldsymbol{c}_i\right) / \left(\boldsymbol{c}_i^T\boldsymbol{S}\boldsymbol{c}_i\right)$, $i = 1, ..., 4$, and for each i, by using WishPack, we draw $T = 50000$ random samples $\boldsymbol{W}_j$ from the Wishart distribution with the true parameters

$$\boldsymbol{S} = \begin{pmatrix} 6 & 2 & -2 \\ 2 & 8 & 1 \\ -2 & 1 & 10 \end{pmatrix}, \ k = 10.5.$$

Thus, according to (1), χ_i^2 should follow $\chi^2(k)$. Figure 2 shows the empirical distributions of χ_i^2 along with their sample means and sample variances. We can see that the sample means are very close to $10.5 = k$ and the sample variances are approximately $21 = 2k$. To have further verification, we also generate the Q-Q plots and conduct the Kolmogorov-Smirnov (KS) test.

The results are given in Figure 3. It is clear that in the Q-Q plots, the points all approximately lie on the 45 degree lines, and the KS test results (in p-values) suggest that χ_i^2 follows $\chi^2(k)$ under the significance level = 0.05. Moreover, we obtain the sample mean of the Wishart matrices

$$\bar{\boldsymbol{W}} \equiv \frac{1}{T}\sum_{j=1}^{T}\boldsymbol{W}_j = \begin{pmatrix} 63.3 & 21.2 & -21.0 \\ 21.2 & 84.1 & 10.4 \\ -21.0 & 10.4 & 105.1 \end{pmatrix} \approx \begin{pmatrix} 63 & 21 & -21 \\ 21 & 84 & 10.5 \\ -21 & 10.5 & 105 \end{pmatrix} = k\boldsymbol{S} = \mathrm{E}\left(\boldsymbol{W}\right).$$

From the above analysis, we can find that Wishpack works nicely; the Wishart random matrices with fractional df are appropriately generated .

3.2 Wishart Matrices with Singularity

When $k < p$, the Wishart distribution $W(k, \boldsymbol{S})$ is singular. The singular Wishart matrix does have practical applications (e.g. Kubokawa and Srivastava, 2008; Kunno,2009), and therefore its generation is also taken into account in WishPack. Nevertheless, unlike the non-singular Wishart case, Bartlett's decomposition cannot be directly applied in the singular situation. For this reason, fractional df is not considered; instead, we develop the function for generating singular Wishart random matrices with integer-valued df k. The procedure is given by:

(0) Any input fractional df will be mapped to its floor, i.e. $k = \lfloor k \rfloor$; $\lfloor \cdot \rfloor$ is the floor function.

(1) Generate a $p \times k$ random matrix $\boldsymbol{B}$, with all entries independently from $\mathrm{N}(0, 1)$.

(2) Take Cholesky decomposition for the scale matrix $\boldsymbol{S} = \boldsymbol{A}\boldsymbol{A}^T$.

(3) $\boldsymbol{W} \equiv \boldsymbol{A}\boldsymbol{B}\boldsymbol{B}^T\boldsymbol{A}^T = (\boldsymbol{A}\boldsymbol{B})(\boldsymbol{A}\boldsymbol{B})^T \sim W(k, \boldsymbol{S})$.

Similar to the study in Section 3.1, we can verify the plausibility of the singular Wishart random matrices by applying (1). All settings and analyzing steps are the same as those in Section 3.1, except that here the true df parameter is $k = 2.5$. Note that, since k can no longer be fractional in the singular case, the df that is actually considered will be $\lfloor k \rfloor = 2$. The results are given in Figure 4 and Figure 5. With similar analyses and discussion, we can conclude that WishPack works well for the generation of Wishart random matrices with singularity. Section 3.1 and 3.2 together illustrate how Wishpack works for the generation of Wishart matrices.

3.3 A Note on Odell-Feiveson's Algorithm

We close this Section with the Odell and Feiveson's (1966; hereafter O&F) algorithm for generating Wishart matrices. This method, also based on Bartlett's decomposition, is popular and has been widely used, e.g. Liu (2001) and the Matlab package "MCMC" by Shera (1998). One should notice that, in O&F's construction, the leading diagonal entry of the Wishart matrix is a chi-square variate with df $N-1$, where N is the number of independent normally distributed random vectors. It is hence clear that the Wishart matrix has $N-1$ df, not N. Consequently, if we would like to employ O&F's method to generate a Wishart matrix from $W(k, \boldsymbol{S})$, in fact we will end up with a matrix from $W(k-1, \boldsymbol{S})$.

To see this, by applying the Matlab package "MCMC", we implement the same analysis as those in Section 3.1 and 3.2. The only change is that the true df is $k = 10$. The results are shown in Figure 6, from which we see that, obviously each χ_i^2 does not follow $\chi^2(10)$. They are more likely to have a $\chi^2(9)$ distribution since the sample means are almost $9 = k-1$ and sample variances are close to $18 = 2(k-1)$. This illustration is to provide an example that we may need to pay extra attention to the use and the explanation of O&F's algorithm.

4 Conclusion and Discussion

This paper demonstrates the importance of the Wishart matrix with fractional df in econometric problems and discusses the generation of Wishart matrices for singular and non-singular cases. Since existing OX functions are unable to generate these types of Wishart random samples, we develop this WishPack in order to meet such needs in Bayesian methodology. The density functions for both Wishart and inverse Wishart distributions are also included.

Srivastava (2003) and Díaz-García (2007) define different measures for the computation of Jacobians for deriving the density of a singular Wishart distribution. In Wishpack, we provide the calculation of the singular Wishart density based on Srivastava's (2003) version. The package contains several header files and can be downloaded at http://www4.ncsu.edu/~yku2/smain.html

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Figure 1: Trace plots of k and $|\boldsymbol{A}|$ from the simulation of A&M's WIC model.

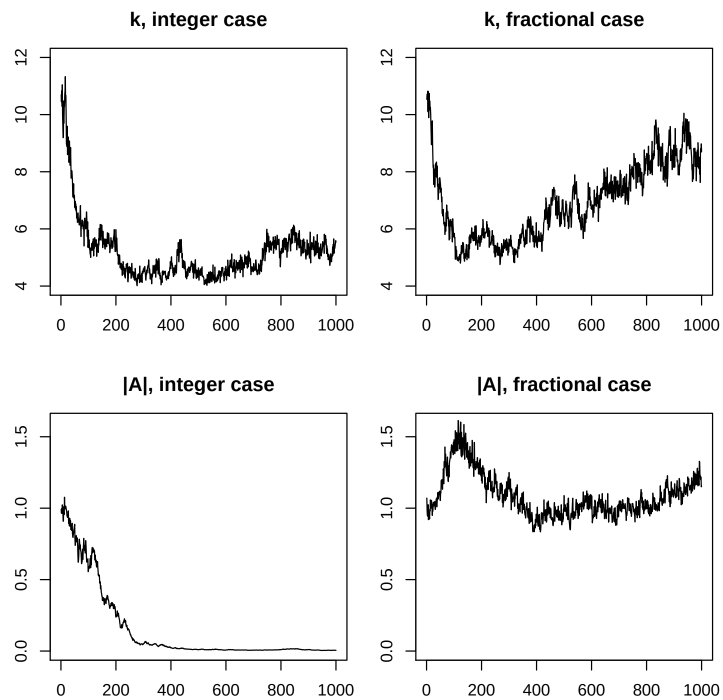

Figure 2: Histograms of χ_1^2, χ_2^2, χ_3^2, and χ_4^2 for non-singular Wishart.

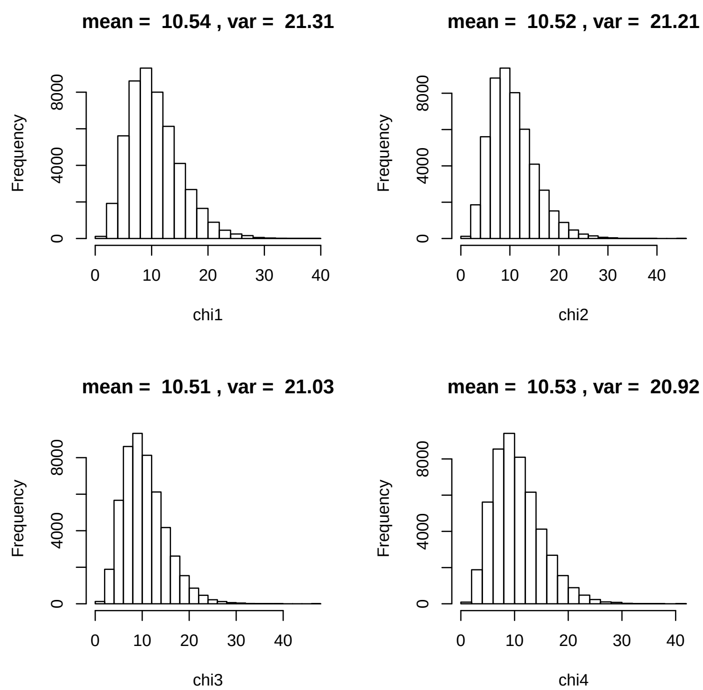

Figure 3: Q-Q plots and K-S test results of χ_1^2, χ_2^2, χ_3^2, and χ_4^2 for non-singular Wishart.

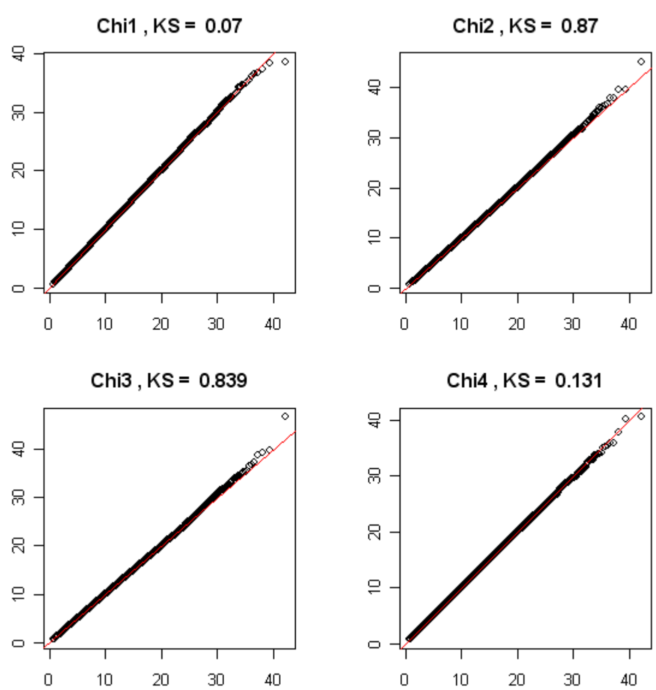

Figure 4: Histograms of χ_1^2, χ_2^2, χ_3^2, and χ_4^2 for singular Wishart.

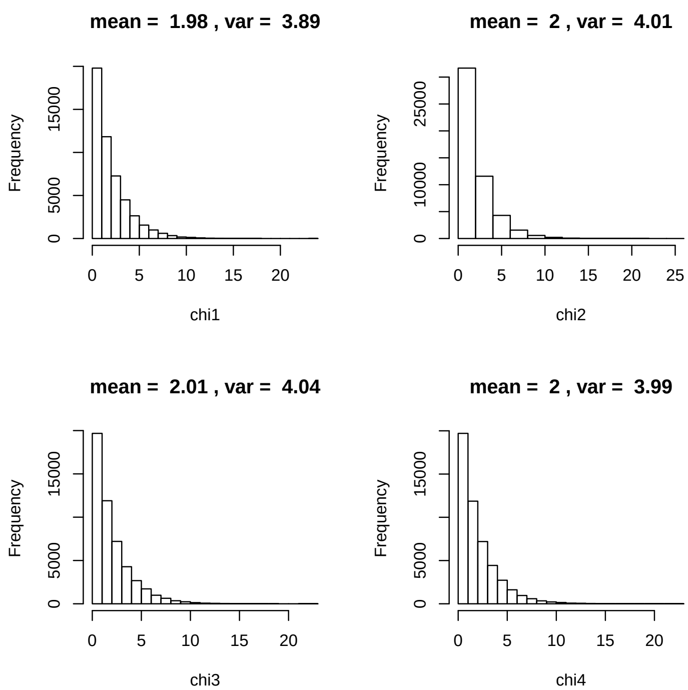

Figure 5: Q-Q plots and K-S test results of χ_1^2, χ_2^2, χ_3^2, and χ_4^2 for singular Wishart.

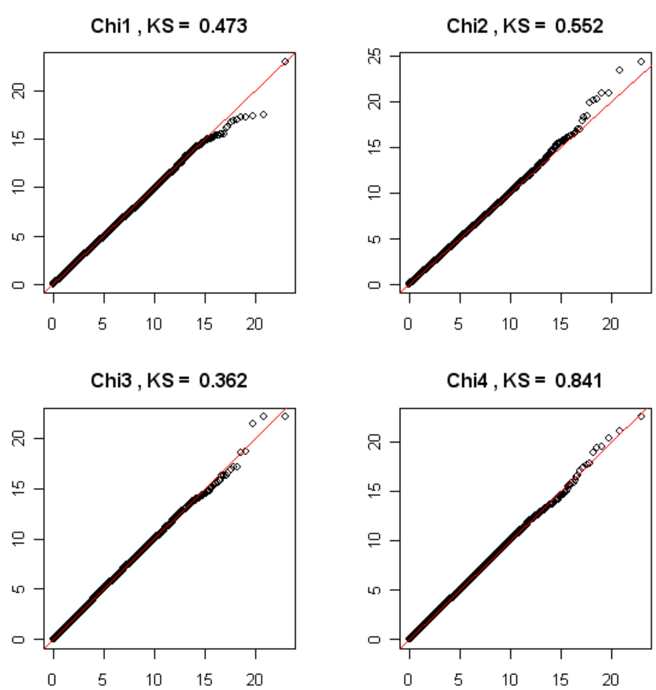

Figure 6: Histograms of χ_1^2, χ_2^2, χ_3^2, and χ_4^2 using incorrectly specified df.