Institute for International Economic Policy Working Paper Series
Elliott School of International Affairs
The George Washington University

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IIEP-WP-2011-14

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September 2011

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Poverty Measurement with Ordinal Data

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Abstract

The Foster, Greer, Thorbecke (1984) class nests several of the most widely used measures in theoretical and empirical work on economic poverty. Use of this general class of measures, however, presupposes a dimension of well-being that, like income, is cardinally measurable. Responding to recent interest in dimensions of well-being where achievements are recorded on an ordinal scale, this paper develops counterparts to the popular FGT measures that are still meaningful when applied to ordinal data. The resulting ordinal FGT measures retain the simplicity of the classical FGT measures and also many of their desirable features, including additive decomposability. This paper also develops ordinal analogues of the core axioms from the literature on economic poverty, and demonstrates that the ordinal FGT measures indeed satisfy these core axioms. Moreover, new dominance conditions, which allow for poverty rankings that are robust with respect to the choice of poverty line, are established. Lastly, the ordinal FGT measures are illustrated using self-reported data on health status in Canada and the United States.

JEL classification: I3, I32, D63, O1

Keywords: poverty measurement, ordinal data, FGT poverty measures, social welfare, dominance conditions

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*We are deeply indebted to James Foster and John Weymark for their advice and support throughout our work on this project. We also wish to thank Buhong Zheng for his insightful comments and suggestions. Finally, the second author gratefully acknowledges the Institute for International Economic Policy of the Elliott School of International Affairs for funding her visit to Vanderbilt University in order to conduct this research.

1 Introduction

In the twenty-five years since it was first introduced, the FGT (Foster, Greer, and Thorbecke 1984) family of measures has become the most widely used class in empirical work on the measurement of poverty. The attractiveness of the FGT measures stems largely from their simple structure, their ease of interpretation, and their sound axiomatic properties. Being defined by two parameters, namely the poverty line ℓ and a scalar measure of poverty aversion α, each member of the FGT class is easily computed as an average of the power function defined by α whose argument is the normalized income shortfall from ℓ. Specific members include the well-known poverty gap, squared poverty gap, and headcount ratio (i.e., the proportion of the population identified as poor).

Use of the general FGT class of measures presupposes a dimension of well-being that, like income, is cardinally measurable. Recently, however, considerable interest has emerged in measures of aggregate deprivation in dimensions of well-being other than income, and, in particular, in dimensions of well-being—for example, health, education, empowerment, and social inclusion, etc.—which are often recorded on an ordinal scale.[1] For example, individual level health data is often available in the form of self-assessments in which survey participants are asked to characterize their health status as either *poor*, *fair*, *good*, *very good*, or *excellent*. While it is common to assign numerical levels such as 1, 2, 3, 4, and 5 to the individual responses and even for such a scale to capture a sense of intensity, except for the headcount ratio, none of the FGT measures are meaningful when applied to such data. As a consequence, "a crucial emerging issue is how to measure poverty when data do not have the characteristics of income, which is typically taken to be cardinal and comparable across persons ... Must we retreat to the headcount ratio [with ordinal data], or can we continue to evaluate the depth or distribution of deprivations—key benefits provided by the higher order FGT measures when the variable is cardinal?"(Foster, Greer, and Thorbecke 2010, p. 516)

In this paper, we introduce a counterpart to the classical FGT class of measures for use with variables measured on an ordinal scale. Our particular construction gives rise to an ordinal counterpart to the FGT class of measures which (i) has sound axiomatic properties; (ii) retains many of the attractive properties of the classical FGT measures

[1] Problems surrounding the measurement of poverty with ordinal data are raised, for example, in Foster, Greer, and Thorbecke (2010) and Alkire and Foster (2011a,b). Allison and Foster (2004) were the first to stress the problems raised by ordinal data in the related context of inequality measurement. See Zheng (2008), Abul Naga and Yalcin (2004), and Madden (2010) for more on the use of ordinal data in this latter context.

(including, for example, additive decomposability); and yet (iii) is without the obvious shortcomings inherent in the application of conventional poverty measures to ordinal data. Importantly, this new class of measures is also endowed with interesting welfare interpretations.

The key insight is that each of the classical FGT measures may be reinterpreted as an individual evaluation based on the expected outcome of an income lottery. In particular, suppose that an individual is completely unaware of her relative position in society and is given the option to accept either an income level drawn at random according to the actual distribution in society, or an income level drawn at random from the poor incomes $[0, \ell]$ according to the power distribution parameterized by α (cf. Figure 1b.). Given her option to choose, we show that for any (α, ℓ) the corresponding FGT measure may be expressed as the *likelihood* or *probability* that an individual would accept the draw from the power distribution parameterized by α rather than the draw from the actual distribution of incomes in society. This novel interpretation of the FGT measures is useful because the probability that an individual would accept a level of achievement drawn at random from the states of poverty (not necessarily income poverty) rather than accept the level of achievement generated from the actual distribution of achievements in society is meaningful irrespective of whether the levels of achievement are measured on an ordinal or cardinal scale.

A particularly important concern that arises when applying measures to ordinal data has to do with the arbitrary nature of the numerical values representing the various levels of achievement. Specifically, because no individual is made better or worse off by an order-preserving transformation applied to the levels of achievement and to the poverty line, any suitable poverty measure (or at least its induced poverty ordering) should be invariant to *any* such transformation.[2] However, it is easy to construct examples in which the ordering of two distributions by virtually any existing measure is reversed when the levels of achievement and poverty line are subjected to a positive monotonic transformation.

The ordinal FGT measures developed in this paper not only avoid this obvious shortcoming, but they also satisfy a number of other attractive properties. In the literature on income poverty, for example, there are a number of core properties generally regarded that any poverty measure should satisfy. A further contribution of the present paper is

[2]This point is emphasized in Alkire and Foster (2011b, p. 306) where they state that "the key requirement for ordinal data is that if the cutoff and variables are changed by a monotonic transformation, the level of poverty must remain unchanged, and the same people must be identified as poor." Also, Allison and Foster (2004) discuss this problem in the context of measuring inequality with self-reported health data.

our development of ordinal analogues of these properties, and our verification of the axioms that are satisfied by our proposed measures. Moreover, we also establish a number of dominance conditions which are analogous to the dominance conditions of Foster and Shorrocks (1988a,b), thereby allowing us to establish poverty rankings that are robust with respect to the choice of poverty line.

Lastly, although our focus in the present paper is on the single dimensional case, it should be emphasized that our measures have immediate implications for the literature on multidimensional poverty measurement. Specifically, because many of the recently proposed multidimensional measures are based on weighted averages of unidimensional FGT measures, they either preclude the use of ordinal variables or, as a consequence of their inclusion, are restricted in their choice of an appropriate unidimensional measure.[3] As noted by Alkire and Foster (2011a, p. 476), for example, a significant challenge that discourages the empirical use of these methodologies is that they "are largely dependent on the assumption that variables are cardinal, when, in fact, many dimensions of interest are ordinal or categorical." The ordinal FGT measures developed in this paper may thus be viewed as complementing the fast-growing literature on multidimensional poverty measurement by enabling users to move beyond the headcount ratio when seeking to incorporate relevant dimensions of well-being that are measured on an ordinal scale.

In the next section we briefly review the FGT class of poverty measures. In this section, we also present an equivalent reformulation of this class which gives rise to an interesting interpretation of the FGT measures as the expected outcome of income lotteries. In Section 3, we exploit this novel reformulation to construct an ordinal counterpart to the classical FGT measures. In Section 4, we develop a formal axiomatic framework for the measurement of poverty with ordinal variables and identify the axioms that are satisfied by our proposed class of measures. Section 4 also develops a social welfare interpretation for this new class of measures and establishes a new set of welfare dominance conditions. Then, in Section 5, we illustrate the application of our poverty measures and related dominance conditions to self-reported health data from the United States and Canada. When applied to this dataset, our measures suggest that there is unambiguously greater ill-health in the United States than in Canada for the bottom 20% of their income distributions. Finally, in Section 6 we present some concluding remarks.

[3] See, for example, Bourguignon and Chakravarty (2003), Duclos, Sahn, and Younger (2006), and Alkire and Foster (2011a), Alkire and Foster (2011b), among others. Notably, only the Alkire and Foster (2011a) methodology permits the inclusion of dimensions of well-being measured on an ordinal scale, however they restrict themselves to the headcount ratio when measuring the contribution of "ordinal" dimensions to overall poverty.

2 The FGT Class of Poverty Measures

Foster, Greer, and Thorbecke (1984), hereafter FGT, introduced the class of poverty measures[4]

$$\pi_{\alpha,\ell}(Y) = \mathbf{E}_Y\left[\left[g(Y;\ell)\right]^{\alpha}\mathbb{1}(Y \leq \ell)\right] \quad \alpha, \ell \in \mathbb{R}_+ \tag{2.1}$$

where Y is income, ℓ is the poverty line, $g(y;\ell) = (\ell - y)/\ell$ is the normalized income shortfall, and $\mathbb{1}(\cdot)$ is the indicator function which is equal to one if the argument is true and equal to zero otherwise.[5] The parameter α which appears in (2.1) controls the degree to which shortfalls are penalized and is therefore often interpreted as an indicator of "poverty aversion." The resulting parametric class nests several of the most widely used measures in both theoretical and empirical studies of poverty measurement. When $\alpha = 0$, for example, the measure reduces to

$$\pi_{0,\ell}(Y) = \mathbf{P}[Y \leq \ell]$$

which is the proportion of the population in poverty, commonly known as the *headcount ratio*. When $\alpha = 1$, the FGT class gives rise to the *poverty gap* measure which has a simple interpretation as the average normalized shortfall. The *squared poverty gap* measure, which reports the average of the *squared* normalized shortfalls, also emerges from the FGT class upon setting $\alpha = 2$. The popularity of these measures stems not only from their ease of interpretation—being based on powers of normalized shortfalls—but also from their sound axiomatic properties which include the attractive additive decomposability and subgroup consistency properties. For a detailed account of the FGT class of measures, including a discussion of the axiomatic foundations and related literature, the reader is referred to Foster, Greer, and Thorbecke (2010).

The classical formulation of the FGT class of measures in terms of "shortfalls" relies critically on a well-defined notion of distance between an individual's level of income and the poverty cut-off. While this formulation is quite natural and appealing in the context of income poverty where the distance between an individual's level of income and the poverty cut-off is not only meaningful but also quite easy to interpret, it does not carry over in any natural way to situations in which the variable of interest has only ordinal

[4] The FGT class of measures is commonly denoted by P_α, where, for example, P_0 is the headcount ratio, P_1 is the poverty gap measure, and P_2 is the squared gap measure. Here, we depart from this conventional notation because of our later usage of probability measures and, hence, our desire to prevent any possible confusion between these objects.

[5] Here, as throughout, we adopt the strong definition of the poor (Donaldson and Weymark 1986) under which the poor consists of all individuals with endowments less than or equal to the poverty line.

significance. Our objective in this section is to develop an alternative formulation of the FGT class of measures which not only gives rise to a novel interpretation of them as the expected outcomes of particular income lotteries, but which also allows us to adapt the FGT class quite naturally to situations where the underlying variables are ordinal.

To begin, let F denote the distribution of income and Y denote a random draw from F. Also, let U be a random variable which is uniformly distributed on $[0, \ell]$ but which is stochastically independent of Y. We then define the survival function of U_ℓ as $G_\ell(y) = \mathbf{P}[U_\ell \geq y]$, and for $\alpha > 0$ we denote by $U_{\alpha,\ell}$ the random variable with survival function $G_\ell^{(\alpha)}(y) \equiv [G_\ell(y)]^\alpha$. Lastly, in the case $\alpha = 0$, we define $U_{0,\ell}$ to be a random variable which has probability mass one at ℓ. With $U_{\alpha,\ell}$ in hand, we introduce the binary random variable

$$\mathrm{I}_{\alpha,\ell} = \mathbb{1}(Y \leq U_{\alpha,\ell})\mathbb{1}(Y \leq \ell). \tag{2.2}$$

Thus, $\mathrm{I}_{\alpha,\ell}$ is equal to one if, and only if, an individual's income generated from F is less than the poverty line *and* less than the realization of the random draw $U_{\alpha,\ell}$ from incomes below the poverty line. The realization of the random variable $\mathrm{I}_{\alpha,\ell}$ may therefore be interpreted as whether the individual would accept the income generated from the poverty incomes according to G_α ($\mathrm{I}_{\alpha,\ell} = 1$) or would decline such an offer in favor of the income generated from the population according to F ($\mathrm{I}_{\alpha,\ell} = 0$).

Of particular interest for the measurement of poverty is the expected value of this lottery, namely

$$\begin{aligned} \mathbf{E}_{Y,U_{\alpha,\ell}}[\mathbb{1}(Y \leq U_{\alpha,\ell})\mathbb{1}(Y \leq \ell)] &= \mathbf{E}_{Y,U_{\alpha,\ell}}[\mathrm{I}_{\alpha,\ell}] \\ &= \mathbf{P}[\mathrm{I}_{\alpha,\ell} = 1], \end{aligned} \tag{2.3}$$

which is the *probability* that an individual would choose to accept the realization of $U_{\alpha,\ell}$ rather than the realization of Y. In order to connect this probabilistic statement to classical approaches to poverty measurement, consider taking expectations of (2.2) first with respect to $U_{\alpha,\ell}$. Doing so yields

$$\mathbf{E}_{Y,U_{\alpha,\ell}}[\mathbb{1}(Y \leq U_{\alpha,\ell})\mathbb{1}(Y \leq \ell)] = \mathbf{E}_Y\left[\left(\frac{\ell - Y}{\ell}\right)^\alpha \mathbb{1}(Y \leq \ell)\right], \tag{2.4}$$

which is the FGT measure $\pi_{\alpha,\ell}(Y)$. Consequently, we see that each member of the FGT class of measures when applied to income has a dual interpretation: first, as the average of (a functional of) normalized income shortfalls; and second, as the probability of accepting the income which is drawn at random from the poor incomes according to $G_\ell^{(\alpha)}$.

The latter interpretation is attractive not only because it is simple to interpret, but also because it remains intact when we reverse the order of expectations to obtain an alternative formulation of the FGT class of measures as

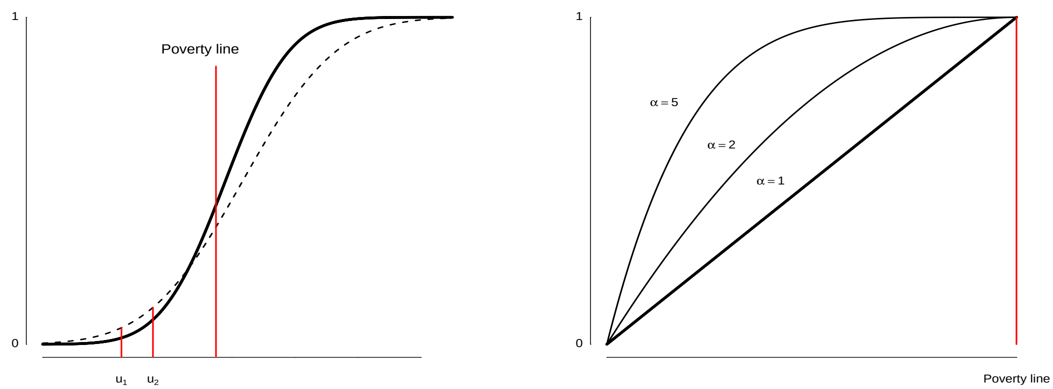
$$\begin{aligned}\mathbf{E}_{Y,U_{\alpha,\ell}}[\mathbb{1}(Y \leq U_{\alpha,\ell})\mathbb{1}(Y \leq \ell)] &= \mathbf{E}_{U_{\alpha,\ell}}[F_{Y|Y\leq\ell}(U_{\alpha,\ell})] \times \mathbf{P}[Y \leq \ell] \\ &= \mathbf{E}_{U_{\alpha,\ell}}[F(U_\alpha)]\end{aligned} \tag{2.5}$$

for $\alpha, \ell \in \mathbb{R}_+$. Equation (2.5) shows that each member of the FGT class is also equivalent to an expected headcount, where the expectation is with respect to a random variable from the set $\{U_{\alpha,\ell}, \alpha \geq 0\}$. That is, imagine using the realization of $U_{\alpha,\ell}$ as the poverty line in the computation of the headcount. The last line of equation (2.5) shows that the FGT measure with parameters (α, ℓ) is in fact equivalent to the average of the headcounts corresponding to the various realizations of $U_{\alpha,\ell}$.

This latter formulation of the FGT class of measures gives rise to an interesting reinterpretation in the spirit of Harsanyi (1953) and Rawls (1999), where one's value judgement on the distribution of income corresponds to a hypothetical situation in which an individual has complete ignorance of his own relative position in society and instead faces an income lottery. When $\alpha = 1$, for example, the income lottery is such that the individual has the same chance of obtaining the lowest income as any other level of income up to the poverty line. The expectation in the last line of (2.5) may therefore be interpreted as the percentile of the income distribution an individual, say, would expect to find herself in if faced with the prospect of drawing her income at random from a $U[0, \ell]$ distribution.

When $\alpha = 2$, the measure may again be interpreted as an expected percentile albeit with the expectation taken with respect to the distribution $1 - G_\ell^{(2)}(y)$. Because the distribution $1 - G_\ell^{(2)}(y)$ as depicted in Figure 1(a) is everywhere above the uniform CDF (i.e., is stochastically dominated by the uniform CDF), there is a lower chance of observing a draw close to the poverty line (i.e., the probability of observing u_2 rather than u_1 in Figure 1(a) has fallen) and hence the corresponding percentiles contribute less to the expected value. This interpretation is also equivalent to regarding $\pi_{2,\ell}$ as a weighted average of the headcount ratios evaluated at all incomes in the interval $[0, \ell]$, but with weights tilted towards the poorest of the poor. Similar interpretations may be obtained for $\alpha > 2$.

In general, rather than framing the FGT measures in terms of average (normalized) shortfalls, the modified formulation obtained in the last line of equation (2.5) frames each



(a) Comparing the values of two hypothetical CDFs at random draws u_1 and u_2

(b) Distributions generating the u_i for various levels of α.

Figure 1: FGT measures as expected outcomes of an income lottery.

FGT measure as an expected value of headcount ratios. Because shortfalls are no longer interpretable in the case of ordinal variables, whereas headcount ratios and even averages of headcount ratios are, the latter interpretation suggests that the FGT class of measures can be adapted so as to apply to ordinal variables.

3 Extending the FGT class of measures to ordinal variables

In this section we seek to extend the FGT class for use with measures of achievement that are recorded on an ordinal scale. We assume that the number of ordered categories of achievement, denoted by n, is exogenously determined, and we let $S = \{y_1, y_2, \ldots, y_n\}$ denote a numerical respresentation where $y_{i+1} > y_i$.[6] Let F record the distribution of achievements for a given population, and denote by Y a random draw from F. Thus, $F(y_1)$ is the proportion of individuals in the population with level of achievement y_1, whereas $F(y_k) - F(y_{k-1})$ is the proportion of individuals in the population with level of achievement y_k. We allow here for the possibility that F apportions zero probability to one or more of the n levels of achievement.

Given a pre-specified poverty line $\ell \in \{y_1, \ldots, y_n\}$, we then define $U_{S,\ell}$ to be a random variable which is uniformly distributed on $S \cap [0, \ell]$. Also, we define the survival function $G_{S,\ell}(u) = P[U_{S,\ell} \geq u]$, and let $U_{\alpha,S,\ell}$ denote a random variable with survival function

[6]The numerical representation is arbitrary apart from the restriction that $0 \leq y_1 < \cdots < y_n < \infty$.

$G^{(\alpha)}_{S,\ell}(y) \equiv [G_{S,\ell}(y)]^{\alpha}$ for $\alpha > 0$. In the case $\alpha = 0$, we define $U_{0,S,\ell}$ to be a random variable which has probability mass one at the poverty line ℓ. With $U_{\alpha,S,\ell}$ in hand, define the random variable

$$\mathrm{I}_{\alpha,\ell} = \mathbb{1}(Y \leq U_{\alpha,S,\ell})\mathbb{1}(Y \leq \ell). \tag{3.1}$$

The random variable $\mathrm{I}_{\alpha,\ell}$ is the analogue of the binary random variable previously defined in (2.2) and has a similar interpretation. Specifically, the variable $I_{\alpha,\ell}$ assumes the value 1 if, and only if, the realization of Y according to F falls at or below the realization of $U_{\alpha,S,\ell}$ according to $G^{(\alpha)}_{S,\ell}$, in which case an individual would be willing to accept the latter as their level of well-being. Consequently, the expected value of $\mathrm{I}_{\alpha,\ell}$ is again the probability that an individual would be willing to accept the realization of $U_{\alpha,S,\ell}$ as their level of well-being over the realization of Y. Expressed in terms of an average of headcounts, we have

$$\begin{aligned} \pi_{\alpha,\ell}(Y) &= \mathbf{E}_{Y,U_{\alpha,S,\ell}}[\mathbb{1}(Y \leq U_{\alpha,S,\ell})\mathbb{1}(Y \leq \ell)] \\ &= \mathbf{E}_{U_{\alpha,\ell}}[F_{Y|Y\leq\ell}(U_{\alpha,S,\ell})] \times \mathbf{P}[Y \leq \ell] \\ &= \mathbf{E}_{U_{\alpha,\ell}}[F(U_{\alpha,S,\ell})]. \end{aligned} \tag{3.2}$$

Because the distribution of $U_{\alpha,S,\ell}$ has its support restricted to the set $S \cap [0,\ell]$, the classical formulation of the FGT class of measures in terms of shortfalls is no longer recovered from (3.2) by simply reversing the order of integration. Notice, however, that if Y is indeed a cardinal random variable with support on the positive reals, then $S \cap [0,\ell] = [0,\ell]$, in which case $U_{\alpha,S,\ell}$ is as defined in (2.2), and the classical FGT class *is* recovered by reversing the order of integration. In this sense, the formulation of $\pi_{\alpha,\ell}$ in (3.2) may be seen to nest the classical FGT measures.

Irrespective of whether the underlying variable is ordinal or cardinal, the modified formulation in (3.2) retains the interpretation as an average headcount (or expected percentile) and also the dual interpretation as an expected poverty status. For example, when $\alpha = 1$, one's value judgement corresponds to the expected percentile in the distribution of levels of achievement in society when an individual has complete ignorance of his own relative position in society and instead faces a lottery with the same chance of obtaining the lowest level of achievement as any other possible level of achievement up to the poverty line.

It is worth emphasizing that the key distinction between (3.2) and the classical formulation of the FGT measures—cf., equation (2.5)—lies in the lottery being over a potentially finite number of states $y_1, \cdots, y_k$ where y_k, is the most preferred state for which someone is still identified as poor, rather than necessarily being over the continuum $[0, \ell]$. It is also important to emphasize that this distinction is not merely technical: when Y is a cardinal variable, such as income, distances between y_1 and y_k, or between y_1 and any $y \in [y_1, y_k]$ are well defined; in contrast, when Y is ordinal, distances are no longer meaningful and, moreover, "levels" of achievement outside of the set $\{y_1, y_2, \ldots, y_n\}$ are not well-defined.

In addition to retaining precisely the same interpretation, the ordinal class of FGT measures as defined in (3.2) also retain the computational simplicity of the classical FGT class of measures. To see this, suppose that Y is an ordinal random variable with distribution $(p_1, y_1; \ldots; p_n, y_n)$. Also, for concreteness, suppose that $\ell = y_k$.[7] Then, enumerating the values of the survival function for any fixed $\alpha > 0$, we obtain

$$\big[G_{S,\ell}(y_{k-j})\big]^\alpha = \left(\frac{j+1}{k}\right)^\alpha, \qquad j = 0, \ldots, k-1.$$

Note that

$$\big[G_{S,\ell}(y_1)\big]^\alpha = 1.$$

The probability mass at y_j is thus given by

$$\mathbf{P}\big[U_{\alpha,S,\ell} = y_j\big] = \left(\frac{k-j+1}{k}\right)^\alpha - \left(\frac{k-j}{k}\right)^\alpha \tag{3.3}$$

for $j \in \{1, \ldots, k\}$. With the probability mass function (3.3) in hand, we immediately obtain the computational formula for the ordinal measure $\pi_{\alpha,\ell}$ for any $\alpha > 0$ as

$$\begin{aligned} \pi_{\alpha,\ell}(Y) &= \sum_{j=1}^{k} F(y_j)\left[\left(\frac{k-j+1}{k}\right)^\alpha - \left(\frac{k-j}{k}\right)^\alpha\right] \\ &= \sum_{j=1}^{k} p_j \left(\frac{k-j+1}{k}\right)^\alpha \end{aligned} \tag{3.4}$$

where $F(y_j) = \sum_{i \le j} p_i$, and the last line follows as a consequence of Abel's partial summation formula (Apostol 1974, p. 194).

[7] Incidentally, because the support of the random variable $U_{\alpha,S,\ell}$ is $\{y_1, \ldots, y_n\} \cap [0, \ell]$, the computational formula is the same for any value of the poverty line in the interval $[y_k, y_{k+1})$.

4 Properties of the Ordinal FGT Class of Measures

A number of intuitively appealing desiderata (axioms) have been put forth in the literature on income poverty; see, for example, Zheng (1997). While some of these axioms carry over without modification to the present context (e.g., the focus, symmetry, and replication invariance axioms), many do not. In fact, a number of axioms which are widely considered to be intuitively appealing in the context of income poverty become rather unappealing when the underlying measure of achievement has only ordinal significance. In this section, we collect—and suitably reformulate when necessary—a set of core axioms for poverty measurement with ordinal variables. We also identify the particular axioms that are satisfied by various members of our proposed parametric class of ordinal FGT measures.

A particularly important issue that arises in the context of measurement with ordinal data has to do with the arbitrary nature of the numerical representation assigned to the levels of achievement. Specifically, because no individual is made better or worse off by an order-preserving transformation applied to the levels of achievement and to the poverty line, any suitable poverty measure (or at least its induced poverty ordering) should be invariant to *any* such transformation. This leads to the following *invariance* axiom:

Axiom 4.1 (Ordinal Invariance)**.** Suppose that ℓ is given and that Y has distribution $(p_1, y_1; p_2, y_2; \ldots; p_n, y_n)$. If $g : \mathbf{R} \to \mathbf{R}$ is monotone increasing, then

$$\pi_{\alpha,\ell}(Y) = \pi_{\alpha,g(\ell)}(g(Y))$$

for every $\ell \in \{y_1, \ldots, y_n\}$ and all $\alpha > 0$.

Ensuring that the ordinal FGT measures satisfy Ordinal Invariance is particularly important in allowing us to apply the measure to ordinal variables as it implies that only changes in individuals' *ranks* in society can produce changes in the measured level of poverty. To see this more clearly, let Y be an ordinal random variable with distribution $(p_1, y_1; \ldots; p_n, y_n)$. Also, for concreteness, consider the case $\alpha = 1$ so that $U_{1,S,\ell}$ is uniformly distributed on $S \cap [0, \ell]$ (again independent of Y), and suppose that $\ell = y_k$.

Then,

$$\begin{aligned}
\pi_{1,\ell}(Y) &= \mathbf{E}_{Y,U_{1,S,\ell}}[\mathbb{1}(Y \leq U_{1,S,\ell})\mathbb{1}(Y \leq \ell)] \\
&= \mathbf{E}_{U_{1,S,\ell}}[F_{Y|Y\leq\ell}(U_{1,S,\ell})] \times \mathbf{P}[Y \leq \ell] \\
&= \frac{\frac{1}{k}\sum_{j=1}^{k}(\sum_{i=1}^{j} p_i)}{\sum_{j=1}^{k} p_j} \times \mathbf{P}[Y \leq \ell] \\
&= \frac{p_1}{k} + \frac{(p_1 + p_2)}{k} + \cdots + \frac{(p_1 + p_2 + \ldots + p_k)}{k}
\end{aligned} \tag{4.1}$$

Now suppose that Y is subjected to a positive monotonic transformation which gives rise to $\tilde{Y} \equiv g(Y)$ with distribution $(p_1, \tilde{y}_1; \ldots; p_n, \tilde{y}_n)$. The corresponding random variate $U_{1,\tilde{S},\tilde{\ell}}$ is then, by definition, uniformly distributed on $[\tilde{y}_1, \tilde{y}_2, \cdots, \tilde{y}_k]$. We then have

$$\begin{aligned}
\pi_{1,m(\ell)}(\tilde{Y}) &= \mathbf{E}_{\tilde{Y},U_{1,\tilde{S},\tilde{\ell}}}[\mathbb{1}(\tilde{Y} \leq U_{1,\tilde{S},\tilde{\ell}})\mathbb{1}(\tilde{Y} \leq g(\ell))] \\
&= \mathbf{E}_{U_{\tilde{Y},1}}[F_{\tilde{Y}|\tilde{Y}\leq\ell}(U_{1,\tilde{S},\tilde{\ell}})] \times \mathbf{P}[g(Y) \leq g(\ell)] \\
&= \frac{\frac{1}{k}\sum_{j=1}^{k}(\sum_{i=1}^{j} p_i)}{\sum_{j=1}^{k} p_j} \times \mathbf{P}[Y \leq \ell] \\
&= \frac{p_1}{k} + \frac{(p_1 + p_2)}{k} + \ldots + \frac{(p_1 + p_2 + \ldots + p_k)}{k}
\end{aligned} \tag{4.2}$$

so that the modified FGT measure is unchanged. Invariance of the general class of ordinal FGT measures to positive monotonic transformations may be verified with only a slight modification of the argument presented above and, hence, the proof is omitted.

If one accepts Ordinal Invariance as a basic axiom in the context of poverty measurement with ordinal data, then outside of our ordinal FGT measures there is currently no option other than to employ the headcount ratio as it is the only measure in the FGT class that does not violate these elementary conditions. However, the well-known shortcomings associated with the headcount ratio as a measure of poverty make the use of this measure in the context of ordinal data most unfortunate. Indeed, it was Sen's (1976) influential paper in which he pointed to the failure of the headcount ratio to satisfy a basic monotonicity axiom and a transfer axiom that gave impetus to the now vast literature on poverty measurement. Fortunately, the class of ordinal FGT measures can avoid these shortcomings. To be more precise, let $y_1, \ldots, y_k$ denote the states of poverty where $y_j \succ y_i$ for any $j > i$ with " $\succ$ " denoting the strict preference relation. The monotonicity axiom in the case of income poverty relates to the response of a poverty measure to an increase or decrease in a single individual's level of income. When translated into statements about the distribution, such an increase or decrease amounts to a *transfer* of probability mass. Consequently, we formulate the ordinal analogue of Sen's (1976) monotonicity axiom as follows:

Axiom 4.2 (Monotonicity)**.** Let ℓ be given and suppose that $(p_1, p_2, \ldots, p_k)$ with $\sum_{i \leq k} p_i \leq 1$ denotes the distribution of Y in each of the corresponding k states of poverty. If $\tilde{Y}$ has the same distribution as Y with the exception that $\tilde{p}_j = p_j - \epsilon$ for some $j \in \{1, \ldots, k\}$ and $\tilde{p}_i = p_i + \epsilon$ for some $i < j$, then $\pi_{\alpha,\ell}(\tilde{Y}) > \pi_{\alpha,\ell}(Y)$.[8]

Similarly, transfer in the context of income poverty relates to the response of a poverty measure to the reallocation of income from one individual to another. When translated into statements about the distribution, a transfer shifts the rank of both individuals and, hence, amounts to two separate transfers of probability mass.

Axiom 4.3 (Transfer)**.** Let ℓ be given and suppose that $(p_1, p_2, \ldots, p_k)$ with $\sum_{i \leq k} p_i \leq 1$ denotes the distribution of Y in each of the corresponding k states of poverty. If $\tilde{Y}$ is generated from the distribution of Y by transfers of probability mass $\tilde{p}_i = p_i - \epsilon$ offset by $\tilde{p}_{i-m} = p_{i-m} + \epsilon$, and $\tilde{p}_j = p_j - \epsilon$ offset by $\tilde{p}_{j+m} = p_{j+m} + \epsilon$, then $\pi_{\alpha,\ell}(\tilde{Y}) > \pi_{\alpha,\ell}(Y)$ whenever $1 \leq i - m < m \leq k$ and $i \leq j$ with $i, j, m \in S \cap [0, \ell]$.

We now show that the ordinal FGT class of measures satisfies Monotonicity when $\alpha > 0$ and Transfer when $\alpha > 1$. To do this we use the formula for $\pi_{\alpha,\ell}$ in the first line of (3.4), and suppose that $\tilde{Y}$ with distribution $(q_1, y_1; \ldots; q_n, y_n)$ is obtained from $(p_1, y_1; \ldots; p_n, y_n)$ by shifting probability mass from p_i to p_{i-m}. More precisely, suppose that $q_i = p_i - \epsilon$ and $q_{i-m} = p_{i-m} + \epsilon$ for some $\epsilon > 0$ with $m + 1 \leq i \leq k$. The difference in the levels of poverty according to the measure $\pi_{\alpha,\ell}$ is then given by

$$\begin{aligned} \pi_{\alpha,\ell}(\tilde{Y}) - \pi_{\alpha,\ell}(Y) &= \sum_{j=1}^{k} \left(\sum_{r=1}^{j} (q_r - p_r) \right) \left[\left(\frac{k-j+1}{k} \right)^{\alpha} - \left(\frac{k-j}{k} \right)^{\alpha} \right] \\ &= \sum_{j=1}^{k} \epsilon \mathbb{1}(i-1 \leq j \leq i) \left[\left(\frac{k-j+1}{k} \right)^{\alpha} - \left(\frac{k-j}{k} \right)^{\alpha} \right] \end{aligned} \tag{4.3}$$

That Monotonicity is satisfied for $\alpha > 0$ is thus verified upon noting that the second line is strictly positive for any $\alpha > 0$. The transfer axiom is closely related to the monotonicity axiom, albeit it involves two transfers of probability mass in equal amounts of, say, ϵ. If

[8] Implicit in our statement of the monotonicity axiom is that that $0 < \epsilon \leq p_j$. Similar restrictions on the transfer of probability mass is maintained in our statement of transfer axiom.

$\tilde{Y}$ denotes the outcome of such a transfer, then the effect on poverty as measured by $\pi_{\alpha,\ell}$ is of the form

$$\begin{aligned}\pi_{\alpha,\ell}(\tilde{Y}) - \pi_{\alpha,\ell}(Y) = &\sum_{r=1}^{k} \epsilon\big[\mathbb{1}(i \le r \le i+m) - \mathbb{1}(j-m \le r \le j)\big] \\ &\times \left[\left(\frac{k-r+1}{k}\right)^{\alpha} - \left(\frac{k-r}{k}\right)^{\alpha}\right],\end{aligned} \tag{4.4}$$

where i, j, and m are integers satisfying $i + m \le j$. Clearly, such transfers have no net effect on poverty when $\alpha = 1$, and are poverty decreasing only when $\alpha > 1$. It follows that the class of poverty measures $\pi_{\alpha,\ell}$ satisfy Transfer whenever α exceeds 1.

The last set of axioms we wish to consider are Additive Decomposability and Subgroup Consistency. Both are satisfied by the classical FGT measures and are widely considered to be important since they "allow poverty to be evaluated across population subgroups in a coherent way" (Foster, Greer, and Thorbecke 2010, p. 492). These axioms are stated formally below:

Axiom 4.4 (Additive Decomposability)**.** Suppose that Y_1 and Y_2 have distributions of achievement P_1 and P_2, and suppose that Y has distribution of achievement $P = \lambda P_1 + (1-\lambda)P_2$. Then,

$$\pi_{\alpha,\ell}(Y) = \lambda \pi_{\alpha,\ell}(Y_1) + (1-\lambda)\pi_{\alpha,\ell}(Y_2).$$

Axiom 4.5 (Subgroup Consistency)**.** Suppose that Y_1 and Y_2 have distributions of achievement P_1 and P_2, and suppose that Y has distribution of achievement $P = \lambda P_1 + (1-\lambda)P_2$. Also, let X_1, X_2, and X have distributions Q_1, Q_2, and $Q = \lambda Q_1 + (1-\lambda)Q_2$, respectively. Then,

$$\pi_{\alpha,\ell}(Y) < \pi_{\alpha,\ell}(X)$$

whenever $\pi_{\alpha,\ell}(Y_1) < \pi_{\alpha,\ell}(X_1)$ and $\pi_{\alpha,\ell}(Y_2) = \pi_{\alpha,\ell}(X_2)$

Verifying that the ordinal FGT measures satisfy Additive Decomposability and Subgroup Consistency is straightforward in light of the linearity of the expectations operator. The details are therefore omitted.

5 Poverty Orderings and Welfare Dominance

In this section, we examine the link between poverty measurement with ordinal data and social welfare evaluations. In our discussion of "social welfare measures" we restrict attention to real-valued non-decreasing functions defined on the outcomes taken by some indicator of well-being in a population. More precisely, let Y denote a variable whose outcomes we take as an indicator of achievement, and suppose that Y has distribution F. Then, in our assessments of social welfare, as in Atkinson (1970), we consider social welfare functions with the additively separable form

$$W_u(Y) = \int u(y) dF(y) \tag{5.1}$$

where u is a "utility" function which is increasing in its argument. The additively separable form implies that social welfare is measured by the expected value of utility, $\mathbf{E}_F[u(Y)]$, where the expectation is with respect to the probabilities attached to individuals in the population attaining various levels of achievement.

In contrast to the global nature of W_u, poverty evaluations are often concentrated on the welfare of those individuals identified as poor. Following Foster and Shorrocks (1988b), for example, we may consider *censored* social welfare functions of the form

$$\begin{aligned} W^*_{u,\ell}(Y) &= \int u(y) dF(y) + u(\ell) \int \mathbb{1}\{y > \ell\} dF(y) \\ &= \mathbf{E}[u(Y)\mathbb{1}\{Y \leq \ell\}] + u(\ell)\mathbf{P}[Y > \ell] \end{aligned} \tag{5.2}$$

where the incomes of the non-poor have been censored at the poverty line. Thus, $W^*_{u,\ell}(Y) > W^*_{u,\ell}(\tilde{Y})$ indicates that (censored) social welfare is greater in the society with distribution of achievement Y than in the society with distribution $\tilde{Y}$ when the poverty line is set equal to ℓ.[9] Owing to the presence of $u(\ell)$ in the second term of (5.2), we note that the censored social welfare function registers an increase in welfare even when the poverty line increases and $\mathbf{P}[Y > \ell]$ remains unchanged. This occurs because the base level of utility assigned to the non-poor rises in response to the adjustment whereas the other terms remain constant as no one moves in or out of poverty.

While both simple and intuitive, social welfare evaluations conducted in this way are subject to two significant shortcomings. First, because there may be several reasonable choices for the poverty line it may not clear which one to focus on. Of course, this is less of an issue when the social welfare orderings generated by the various poverty lines are in agreement. Second, the ordering depends on the choice of utility function, but up to an order preserving transformation, this choice is essentially arbitrary. This concern

[9] See, for example, Foster and Shorrocks (1988b) on welfare comparisons based on censored distributions, and also Ravallion (1994) for a discussion of welfare-based poverty evaluations.

is mitigated if there is a consensus—that is, when $W^*_{u,\ell}(Y) > W^*_{u,\ell}(\tilde{Y})$ holds not just for a single u but for all $u \in \mathcal{U}$ where $\mathcal{U}$ denotes the collection (or a sub-collection) of all monotone increasing transforms of u, and for a broad range of reasonable poverty lines.

Specializing our discussion now to the case of ordinal variables, let F denote the distribution of achievements over n possible levels labeled $y_1 < y_2 < \cdots < y_n$. Without loss of generality, $y_j > y_{j-1}$indicates that state j is strictly preferred to state $j-1$. An important implication of the ordinal nature of Y is that the only consistent welfare functions are those which depend only on the rankings of the various levels of achievement.[10] This motivates considering the class of social welfare functions

$$W_u(Y) = \mathbf{E}_F[v\big(L(Y)\big)], \tag{5.3}$$

where $L(y)$ is the discrete uniform distribution on $\{y_1, \ldots, y_n\}$ and $v(\cdot)$ is strictly increasing in its argument. Notice that $L(y)$ simply maps the levels of achievement to their respective (normalized) ranks so that, for example, y_j maps to j/n.[11] Importantly, with this definition, only rank (up to a scale factor) enters as an argument into the utility function and, hence, the social welfare function itself is invariant to order-preserving relabelings of the numerical values assigned to the various levels of achievements. Because the inverse cdf $L^{-1}(y)$ is strictly increasing, the social welfare function

$$W_u(Y) = \mathbf{E}_F[Y] \tag{5.4}$$

is obtained from (5.3) by setting $v(x) = L^{-1}(x)$. In other words, the utility function $u(y) = y$ is nested within the general class of functions $v(L(y))$ on which social welfare evaluations are based. A particularly interesting subclass of social welfare functions is also obtained from (5.3) by setting $v_\alpha(x) = x^{1/\alpha}$. In this case, for example, with $\alpha = 1$, marginal utility is constant (i.e., $u(y_j) - u(y_{j-1}) = 1/k$) irrespective of the values assigned to the y_j (provided of course that $y_j > y_{j-1}$). On the other hand, if $\alpha > 1$, marginal utility is decreasing in y, thereby reflecting an aversion to poverty.

Our goal now is to establish a set of dominance conditions which enable us to make unambiguous statements concerning social welfare rankings (or, equivalently, poverty orderings) of two distributions based on the general class of social welfare functions of the

[10]We say a social welfare function is *consistent* if it preserves the social ranking of alternatives under positive monotonic transformations applied to the outcomes.

[11]Because any increasing function u can be recovered as the composition of $v(\cdot)$ and $L(\cdot)$, the function $L(\cdot)$ itself imposes no restrictions on the class of social welfare functions. Rather, the admissible class of social welfare functions is determined entirely by the admissible class of $v(\cdot)$.

form (5.3). Towards this end, let P and Q denote probability distributions associated with the random levels of achievement Y and X, both defined on the outcome space $\{y_1, y_2, \ldots, y_n\}$.

Theorem 5.1. *For fixed distributions of achievement P and Q, the poverty index $\pi_{0,\ell}$ satisfies*

$$\pi_{0,\ell}(Y) - \pi_{0,\ell}(X) \leq 0, \tag{5.5}$$

for all $\ell \in \{1, \ldots, n-1\}$ if, and only if,

$$\mathbf{E}_P[v(L(Y))] \geq \mathbf{E}_Q[v(L(X))] \tag{5.6}$$

for all $v : [0,1] \to \mathbf{R}$ such that $v'(x) > 0$.

Proof. First, write

$$\begin{aligned} \mathbf{E}_P[v(L(Y))] - \mathbf{E}_Q[v(L(X))] &= \sum_{i=1}^{n}(p_i - q_i)v(L(y_i)) \\ &= \sum_{i=1}^{n-1}\left(\sum_{j=1}^{i}(p_j - q_j)\right)[v(L(y_i)) - v(L(y_{i+1}))], \end{aligned} \tag{5.7}$$

where the second line follows from Abel's partial summation formula. Now, assume that (5.5) holds, in which case we have

$$\sum_{j=1}^{\ell}(p_j - q_j) \leq 0 \text{ for all } \ell = 1, \ldots, n-1,$$

by definition of $\pi_{0,\ell}$. That (5.7) is non-negative then follows as a consequence of the strict monotonicty of v which ensures that

$$v(L(y_i)) - v(L(y_{i+1})) < 0 \text{ for all } 1 \leq i \leq n-1.$$

To prove the converse, we proceed by way of contradiction. Thus, suppose that (5.6) holds for all $v : [0,1] \to \mathbf{R}$ such that $v'(x) > 0$, but that $\sum_{i=1}^{\bar{\ell}}(p_j - q_j) > \epsilon$ for some

$\bar{\ell} \in \{1, \ldots, n-1\}$ and some $\epsilon > 0$. Rewriting (5.7) to isolate the $\bar{\ell}^{\text{th}}$ term, we obtain

$$\begin{aligned}
\mathbf{E}_P[v(L(Y))] - \mathbf{E}_Q[v(L(X))] &= \sum_{i=1}^{\bar{\ell}-1} \left(\sum_{j=1}^{i} (p_j - q_j) \right) [v(L(y_i)) - v(L(y_{i+1}))] \\
&+ \sum_{i=\bar{\ell}+1}^{n-1} \left(\sum_{j=1}^{i} (p_j - q_j) \right) [v(L(y_i)) - v(L(y_{i+1}))] \\
&+ \sum_{j=1}^{\bar{\ell}} (p_j - q_j) [v(L(y_{\bar{\ell}})) - v(L(y_{\bar{\ell}+1}))] \\
&\leq \sum_{i:1 \leq i \leq n-1, i \neq \bar{\ell}} [v(L(y_{i+1})) - v(L(y_i))] - \epsilon [v(L(y_{\bar{\ell}+1})) - v(L(y_{\bar{\ell}}))]
\end{aligned} \tag{5.8}$$

The last line in (5.8) can be made negative whenever v is chosen such that

$$\frac{\sum_{i:1 \leq i \leq n-1, i \neq \bar{\ell}} [v(L(y_{i+1})) - v(L(y_i))]}{[v(L(y_{\bar{\ell}+1})) - v(L(y_{\bar{\ell}}))]} < \epsilon, \tag{5.9}$$

and, because such a utility function v can be constructed within the class of monotone utility functions satisfying $v'(x) > 0$, we obtain the desired contradiction of our original supposition. □

From Theorem 5.1, we see that if $\pi_{0,\ell}(Y) < \pi_{0,\ell}(X)$ for all poverty lines ℓ, then P ranks above Q in terms of social welfare according to all increasing social welfare functions that may be expressed in the form of (5.3). Conversely, if P is ranked above Q according to all increasing social welfare functions expressible in the form of (5.3), then, according to $\pi_{0,\ell}$, there is higher poverty in Q than in P for any given poverty line ℓ. The welfare dominance condition at $\alpha = 0$ is thus equivalent to the standard first-order stochastic dominance condition; see, for example, Atkinson (1987) and Foster and Shorrocks (1988a).

An immediate and interesting corollary of Theorem 5.1 is that the ordering of a *censored* social welfare evaluation, based on any member of the class of social welfare functions defined in (5.3), satisfies $W^*_{u,\bar{\ell}}(Y) \geq W^*_{u,\bar{\ell}}(X)$ whenever $\pi_{0,\ell}(X) \leq \pi_{0,\ell}(Y)$ for every $\ell = 1, ..., \bar{\ell}$. Consequently, we see that rather general statements concerning social welfare can still be obtained in situations where a poverty ranking holds only over a restricted range of potential poverty lines. Moreover, because $-1 \times W^*_{u,\bar{\ell}}$ constructed from a given increasing social welfare function $W_{u,\ell}$ may be interpreted as a poverty index, we see that the ranking $\pi_{0,\ell}(Y) \leq \pi_{0,\ell}(X)$ for every $\ell = 1, ..., \bar{\ell}$ implies that all poverty indices of the form $-1 \times W^*_{u,\bar{\ell}}$ agree with $\pi_{0,\bar{\ell}}$ in their assessment of relative poverty .

We now examine the relevant dominance conditions for $\alpha = 1$ and $\alpha = 2$. The results are provided in Theorems 5.2 and 5.3 below:

Theorem 5.2. *For fixed distributions of achievement P and Q, the poverty index $\pi_{1,\ell}$ satisfies*

$$\pi_{1,\ell}(Y) - \pi_{1,\ell}(X) \leq 0, \tag{5.10}$$

for all $\ell \in \{1, \ldots, n-1\}$ if, and only if,

$$\mathbf{E}_P[v(L(Y))] \geq \mathbf{E}_Q[v(L(X))] \tag{5.11}$$

for all $v : [0,1] \to \mathbf{R}$ such that $v'(x) > 0$ and $v''(x) \leq 0$ for all $x \in (0,1)$.

Proof. Applying Abel's partial summation formula to (5.7) yields

$$\begin{aligned}\mathbf{E}_P[v(L(Y))] - \mathbf{E}_Q[v(L(X))] &= \sum_{i=1}^{n-1}\Big(\sum_{j=1}^{i}(p_j - q_j)\Big)[v(L(y_{n-1})) - v(L(y_n))] \\ &\quad + \sum_{i=1}^{n-2}\Big(\sum_{k=1}^{i}\sum_{j=1}^{k}(p_j - q_j)\Big)[\Delta_i - \Delta_{i+1}],\end{aligned} \tag{5.12}$$

where

$$\Delta_i = v(L(y_i)) - v(L(y_{i+1})).$$

Now, given $v : [0,1] \to \mathbf{R}$ such that $v'(x) > 0$ and $v''(x) \leq 0$ for all $x \in (0,1)$, it must be the case that the difference in expected utility is non-negative whenever

$$\sum_{i=1}^{\ell}\sum_{j=1}^{i}(p_j - q_j) \leq 0 \text{ for } 1 \leq \ell \leq n.$$

For any ℓ, however, we have

$$\begin{aligned}\sum_{i=1}^{\ell}\sum_{j=1}^{i}(p_j - q_j) &= \sum_{i=1}^{\ell}(\ell - i + 1)(p_i - q_i) \\ &= \ell[\pi_{1,\ell}(P) - \pi_{1,\ell}(Q)]\end{aligned} \tag{5.13}$$

thereby showing that (5.10) being satisfied for all poverty lines is sufficient for establishing (5.11) for all increasing (weakly) concave v.

In order to prove the converse, we again proceed by way of contradiction. Thus, suppose that (5.11) holds for all increasing and (weakly) concave v, but that $\sum_{i=1}^{\bar{\ell}}\sum_{j=1}^{i}(p_j - q_j) > \epsilon$ for some $\bar{\ell} \in \{1, \ldots, n-1\}$. We will derive a contradiction in the case $\bar{\ell} = n-1$, and note only that the cases $\bar{\ell} < n-1$ can be handled analogously. In particular, suppose that $v(x) = x$. This utility function is strictly increasing and weakly concave, and yet the corresponding value of (5.11) is negative because of (5.12) and the fact that

$$\sum_{i=1}^{n-1}\left(\sum_{j=1}^{i}(p_j - q_j)\right)[v(L(y_{n-1})) - v(L(y_n))] < \epsilon[v(L(y_{n-1})) - v(L(y_n))] < 0$$

and $\Delta_i - \Delta_{i+1} = 0$ for $i = 1, \ldots, n-2$. □

Theorem 5.2 shows that the poverty ordering $\pi_{1,\ell}$ completely characterizes the social welfare ordering of any two distributions in the sense that a global (i.e., over all possible ℓ) poverty ranking, which is based on $\pi_{1,\ell}$, corresponds to a social welfare ordering that is unanimous among all social welfare functions constructed from strictly increasing and weakly concave v.[12] Moreover, because $W^*_{u,\bar{\ell}}(Y)$ is equivalent to $W_u(\tilde{Y})$ when $\tilde{Y}$ is the random variable Y censored at $\bar{\ell}$, it follows from Theorem 5.2 that the welfare rankings based on the increasing and weakly concave welfare functions $W^*_{u,\bar{\ell}}$ are all in agreement provided that $\pi_{1,\bar{\ell}}$, when applied to the variables censored at $\bar{\ell}$, is consistent in its ranking at all poverty lines up to $\bar{\ell}$.

While the $\pi_{1,\ell}$ measure characterizes social welfare rankings over an extremely broad class of social welfare functions, the welfare ordering imposes weak concavity on the class of admissible utility functions. As noted by Deaton and Paxson (1998), the assumption that social welfare is concave in consumption or income is less tenuous than assuming that social welfare is concave in an ordinal measure of achievement. Nevertheless, in their consideration of self reported health status, Deaton and Paxson (1998) find evidence in support of such a concavity assumption. Restricting attention to the class of weakly concave social welfare functions may thus be justified in certain contexts, though, of course, only after careful consideration of the specific context and nature of the ordinal variable being examined.

We now turn our attention to the case $\alpha = 3$:

Theorem 5.3. *Suppose that*

$$\sum_{i=1}^{\ell}(\ell - i + 1)(\ell - i + 2)(p_i - q_i) \leq 0 \tag{5.14}$$

[12] Yalonetzky (2011) independently derives conditions for second-order stochastic dominance. However, his paper is concerned with developing statistical test procedures. We also recently became aware of the work of Spector, Leshno, and Horin (1996), who also consider stochastic dominance conditions for ordinal variables, though in the context of ordering risky prospects based on the expected utility criterion.

for all $\ell = 1, \ldots, n-2$, *and that* $\pi_{1,n-1}(Y) \leq \pi_{1,n-1}(X)$. *Then,*

$$\mathbf{E}_P[v(L(Y))] \geq \mathbf{E}_Q[v(L(X))] \tag{5.15}$$

for all $v : [0,1] \to \mathbf{R}$ *such that* $v'(x) > 0$, $v''(x) \leq 0$, *and* $v'''(x) \geq 0$ *for all* $x \in (0,1)$.

Proof. The proof again relies on an expansion using Abel's partial summation formula. In particular, applying the partial summation formula to (5.12) we obtain

$$\begin{aligned}\mathbf{E}_P[v(L(Y))] - \mathbf{E}_Q[v(L(X))] &= \sum_{i=1}^{n-1}\Big(\sum_{j=1}^{i}(p_j - q_j)\Big)\Delta_{n-1} \\ &+ \sum_{i=1}^{n-2}\Big(\sum_{k=1}^{i}\sum_{j=1}^{k}(p_j - q_j)\Big)[\Delta_{n-2} - \Delta_{n-1}] \\ &+ \sum_{i=1}^{n-3}\Big(\sum_{m=1}^{i}\sum_{k=1}^{m}\sum_{j=1}^{k}(p_j - q_j)\Big)[(\Delta_i - \Delta_{i+1}) - (\Delta_{i+1} - \Delta_{i+2})]\end{aligned} \tag{5.16}$$

where Δ_i is the same as defined above in the proof of Theorem 5.2. That $\pi_{1,n-1}(P) \leq \pi_{1,n-1}(Q)$ is sufficient to ensure that the first line of (5.16) is non-negative is immediate upon noting that v is strictly increasing. To complete the proof, first notice that we can write the triple sum as

$$\sum_{i=1}^{\ell}\sum_{k=1}^{i}\sum_{j=1}^{k}(p_j - q_j) = \frac{1}{2}\sum_{i=1}^{\ell}(\ell - i + 1)(\ell - i + 2)(p_i - q_i) \tag{5.17}$$

Then, because $v : [0,1] \to \mathbf{R}$ satisfies $v'(x) > 0$, $v''(x) \leq 0$, and $v'''(x) \geq 0$ for all $x \in (0,1)$, a sufficient condition for non-negativity of (5.16) is that the r.h.s. of (5.17) not be positive for any choice of ℓ; that is, we have established that, together with $\pi_{1,n-1}(P) \leq \pi_{1,n-1}(Q)$,

$$\sum_{i=1}^{\ell}(\ell - i + 1)(\ell - i + 2)(p_i - q_i) \leq 0 \text{ for all } \ell = 1, \ldots, n-2$$

is sufficient to obtain the desired result. □

In contrast to the first two cases, Theorem 5.3 shows that it is a hybrid of the ordinal

FGT measures at $\alpha = 1$ and $\alpha = 2$, namely[13]

$$\tilde{\pi}_\ell = 1/2\pi_{2,\ell} + 1/2\pi_{1,\ell}, \tag{5.18}$$

that characterizes the social welfare ranking of two distributions based on any social welfare function constructed from an underlying utility function $v : [0,1] \to \mathbf{R}$ such that $v'(x) > 0$, $v''(x) \leq 0$, and $v'''(x) \geq 0$ for all $x \in (0,1)$. In fact, $\tilde{\pi}$ as defined in (5.18) is simply an equally weighted average of the two poverty measures at $\alpha = 1$ and $\alpha = 2$, which apportions less weight to the poorest of the poor than does $\pi_{2,\ell}$, but greater weight than does $\pi_{1,\ell}$. Theorem 5.3 therefore suggests that, beyond $\alpha = 1$, there is a potential role for convex combinations of ordinal FGT measures that give rise to new poverty measures with profiles of "poverty aversion" that are intermediate among the $\pi_{\alpha,\ell}$ measures at integer-valued α's. Consideration of ordinal poverty measures constructed as convex combinations of the $\pi_{\alpha,\ell}$ measures raises some interesting and important questions. However, such considerations are beyond the scope of the present paper.

6 Empirical Illustration

We now illustrate the ordinal FGT measures using self-reported health statuses in Canada and the United States from the Joint Canada/United States Survey of Health (JCUSH). In these surveys, approximately 3,500 Canadian and 5,200 U.S. residents rated their individual health as either *poor*, *fair*, *good*, *very good*, or *excellent*.[14] Due to the complex sampling design and over-sampling of certain populations, sampling weights have been appended to the survey data by the Centers for Disease Control and Prevention and Statistics Canada to render the samples representative of their respective populations. We use these sampling weights in our subsequent analysis.

We apply the ordinal FGT measures to examine health deprivation or health poverty, as well as to examine health poverty when the population is decomposed by income quintiles. We begin by considering the headcount ratios ($\alpha = 0$) in each country and at various cut-offs.[15] As can be seen in Table 1, more U.S. residents as a proportion of the

[13] This follows upon decomposing the product $(\ell - i + 1)(\ell - i + 2)$ that appears in the RHS of equation (5.17) as $(\ell - i + 1)^2 + (\ell - i + 1)$.

[14] The survey participants were asked "In general, would you say your health is: *poor*, *fair*, *good*, *very good*, or *excellent*?" See Allison and Foster (2004), and references therein, for a discussion and a literature review of the role of self reported health data as a predictor of mortality and overall health.

[15] A similar analysis using the headcount ratio to look at poverty with self-reported health data was performed in Allison and Foster (2004).

Table 1: Headcount ($\alpha = 0$) and Ordinal FGT measure ($\alpha = 1$) decomposed by income quintiles. Estimates based on 2,960 and 3,815 Canadian and U.S. respondents from the 2003 Joint Canada/United States Survey of Health.

Measure	Country	Income Quintile	Cut-Off 1	2	3	4	5
Headcount, $\alpha = 0$	USA	All	0.037	0.136	0.398	0.732	1.000
		1	0.087	0.243	0.532	0.790	1.000
		2	0.052	0.202	0.497	0.783	1.000
		3	0.014	0.096	0.398	0.750	1.000
		4	0.012	0.057	0.288	0.684	1.000
		5	0.008	0.056	0.247	0.644	1.000
	CAN	All	0.032	0.111	0.384	0.757	1.000
		1	0.051	0.154	0.495	0.820	1.000
		2	0.054	0.170	0.467	0.810	1.000
		3	0.032	0.104	0.362	0.756	1.000
		4	0.013	0.084	0.337	0.746	1.000
		5	0.005	0.042	0.249	0.651	1.000
Ordinal FGT, $\alpha = 1$	USA	**All**	**0.037**	**0.087**	**0.191**	**0.326**	**0.461**
		1	**0.087**	**0.165**	**0.287**	**0.413**	**0.530**
		2	0.052	0.127	0.251	0.384	0.507
		3	0.014	0.055	0.170	0.315	0.452
		4	0.012	0.034	0.119	0.260	0.409
		5	0.008	0.032	0.104	0.238	0.391
	CAN	**All**	**0.032**	**0.072**	**0.176**	**0.321**	**0.456**
		1	**0.051**	**0.102**	**0.233**	**0.380**	**0.504**
		2	0.054	0.112	0.230	0.375	0.500
		3	0.032	0.068	0.166	0.314	0.451
		4	0.013	0.049	0.145	0.295	0.436
		5	0.005	0.023	0.098	0.236	0.389

population report their health as being less than or equal to *poor*, *fair*, or *good*, than do in Canada. On the other hand, Canadians are less likely than U.S residents to rate their health status as *excellent*.

For $\alpha = 1$, the ordinal FGT measures suggest that health status in the U.S. is worse than in Canada for every cutoff. Perhaps more interestingly, the decomposition by income quintiles demonstrates that the greatest contribution to the disparity between the two countries occurs at the lowest income quintile. In other words, the disparity in health statuses between the two countries is greatest at the bottom income quintile where the self-reported health statuses of income poor U.S. residents are being compared to self-reported health statuses of income poor Canadians. The $\alpha = 1$ case provides us with more insight into the distribution of the poor than the headcount ratios do by themselves. Such insight may be helpful to policymakers when designing and targeting their health-care policies.

These data can also be used to illustrate the simple interpretation of the ordinal FGT measures provided above. For example, if we focus on the first income quintile and a cutoff of 2, we observe that the FGT measures when $\alpha = 1$ are 0.165 in the U.S. and 0.102 in Canada. Consequently, we have that 165 out of every 1,000 U.S. residents would prefer an equiprobable lottery from the two lowest states of health rather than draw their health status from the actual distribution of health in society. In contrast, only 102 out of every 1,000 Canadian residents would prefer the equiprobable two-state lottery over the random draw from the societal distribution in Canada.

7 Concluding Remarks

In this paper, we have considered the issue of poverty measurement in the context of ordinal data. In particular, we have developed a new class of ordinal measures that retains many of the attractive properties of the FGT class of measures (including, for example, additive decomposability) and yet is without the obvious shortcomings inherent in the application of conventional poverty measures to ordinal data. Additionally, we have established a set of dominance conditions, which enable us to give welfare interpretations to our measures, and also allow us to obtain poverty rankings that are robust to the choice of poverty lines.

Applying the ordinal FGT measures to ordinal data on health statuses in Canada and the United States, we find that U.S. residents are health poor relative to Canadians at lower cut-offs when $\alpha = 0$. When α is increased to 1, however, the ranking is robust to the choice of poverty line since U.S. residents are found to be health poor relative to Canadians at all cut-offs. Interestingly, this latter finding suggests that the proportion of residents who would prefer an equiprobable lottery from the five states of health rather than draw their health status from their actual distribution of health in society is greater in the United States than in Canada. A decomposition by income quintiles also demonstrates that the greatest contribution to the disparity in health statuses between the two countries is greatest at the bottom income quintile where the self-reported health statuses of income poor U.S. residents are being compared to self-reported health statuses of income poor

Canadians. Overall, our analysis suggests that the ordinal FGT measures for $\alpha \geq 1$ provide us with considerably more insight into the distribution of the poor than the headcount ratios do by themselves.

Finally, our focus throughout has been on the development of univariate measures of deprivation. However, given that ordinal variables are so frequently encountered in discussions of multidimensional poverty, it is of considerable interest to examine the potential for integrating the ordinal FGT measures in a multidimensional framework. For example, the ordinal FGT measures can be seamlessly integrated with the Alkire and Foster (2011a) class of multidimensional measures, thereby giving rise to a richer class of measures that possesses even further attractive properties (Bennett and Hatzimasoura 2011).

References

Abul Naga, R. H., and T. Yalcin (2004): "Inequality Measurement for Ordered Response Health Data," *Journal of Health Economics*, 23, 1614–1625.

Alkire, S., and J. Foster (2011a): "Counting and Multidimensional Poverty Measurement," *Journal of Public Economics*, 95, 476–487.

——— (2011b): "Understandings and Misunderstandings of Multidimensional Poverty Measurement," *Journal of Economic Inequality*, 9, 289–314.

Allison, R. A., and J. E. Foster (2004): "Measuring Health Inequality using Qualitative Data," *Journal of Health Economics*, 27, 505–524.

Apostol, T. M. (1974): *Mathematical Analysis*. Addison-Wesley, Reading, MA, second edn.

Atkinson, A. B. (1970): "On the Measurement of Inequality," *Journal of Economic Theory*, 2(3), 244–263.

Atkinson, A. B. (1987): "On the Measurement of Poverty," *Econometrica*, 55, 749–64.

Bennett, C. J., and C. Hatzimasoura (2011): "Multidimensional Poverty Measurement and the Depth of Poverty," *Work in progress*.

Bourguignon, F., and S. R. Chakravarty (2003): "The Measurement of Multidimensional Poverty," *Journal of Economic Inequality*, 1, 25–49.

Deaton, A. S., and C. H. Paxson (1998): "Aging and Inequality in Income and Health," *The American Economic Review*, 88(2).

Donaldson, D., and J. A. Weymark (1986): "Properties of Fixed-Population Poverty Indices," *International Economic Review*, 27, 667–688.

Duclos, J.-Y., D. E. Sahn, and S. D. Younger (2006): "Robust Multidimensional Poverty Comparisons," *The Economic Journal*, 116, 943–968.

Foster, J., J. Greer, and E. Thorbecke (1984): "A Class of Decomposable Poverty Measures," *Econometrica*, 52, 761–766.

——— (2010): "The Foster-Greer-Thorbecke (FGT) Poverty Measures: 25 Years Later," *Journal of Economic Inequality*, 8, 491–524.

Foster, J. E., and A. F. Shorrocks (1988a): "Poverty Orderings," *Econometrica*, 56, 173–177.

——— (1988b): "Poverty Orderings and Welfare Dominance," *Social Choice and Welfare*, 5, 179–198.

Harsanyi, J. C. (1953): "Cardinal Utility in Welfare Economics and in the Theory of Risk-taking," *Journal of Political Economy*, 61, 434.

Madden, D. (2010): "Ordinal and Cardinal Measures of Health Inequality: An Empirical Comparison," *Health Economics*, 19, 243–250.

Ravallion, M. (1994): "Measuring Social Welfare With and Without Poverty Lines," *The American Economic Review*, 84, 359–364.

Rawls, J. (1999): *A Theory of Justice*. Oxford University Press, Oxford.

Sen, A. (1976): "Poverty: An Ordinal Approach to Measurement," *Econometrica*, 44, 219–231.

Spector, Y., M. Leshno, and M. B. Horin (1996): "Stochastic Dominance in an Ordinal World," *European Journal of Operational Research*, 93, 620–627.

Yalonetzky, G. (2011): "Stochastic Dominance with Ordinal Variables: Conditions and a Test," *(Forthcoming) Econometric Reviews*.

Zheng, B. (1997): "Aggregate Poverty Measures," *Journal of Economic Surveys*, 11, 123–162.

——— (2008): "Measuring Inequality with Ordinal Data: A Note," Inequality and Opportunity: Papers from the Second ECINEQ Society Meeting (Research on Economic Inequaltiy, Volume 16), pp. 177–188. Emerald Group Publishing Limited.