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Ana Fostel
John Geanakoplos
George Washington University

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Institute for International Economic Policy
1957 E St. NW, Suite 502
Voice: (202) 994-5320
Fax: (202) 994-5477
Email: **iiep@gwu.edu**
Web: **www.gwu.edu/~iiep**

Leverage and Default in Binomial Economies: A Complete Characterization.

Ana Fostel[*] John Geanakoplos[‡]

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Abstract

Our paper provides a complete characterization of leverage and default in binomial economies with financial assets serving as collateral. First, our Binomial No-Default Theorem states that any equilibrium is equivalent (in real allocations and prices) to another equilibrium in which there is no default. Thus actual default is irrelevant, though the potential for default drives the equilibrium and limits borrowing. This result is valid with arbitrary preferences and endowments, arbitrary promises, many assets and consumption goods, production, and multiple periods. We also show that the no-default equilibrium would be selected if there were the slightest cost of using collateral or handling default. Second, our Binomial Leverage Theorem shows that equilibrium *LTV* for non-contingent debt contracts is the ratio of the worst-case return of the asset to the riskless rate of interest. Finally, our Binomial Leverage-Volatility theorem provides a precise link between leverage and volatility.

Keywords: Endogenous Leverage, Default, Collateral Equilibrium, Financial Asset, Binomial Economy, VaR, Diluted Leverage, Volatility.

JEL Codes: D52, D53, E44, G01, G11, G12.

[*]George Washington University,Washington, DC. New York University. NY. Email: afostel@gwu.edu.

[†]Yale University, New Haven, CT and Santa Fe Institute, Santa Fe, NM. Email: john.geanakoplos@yale.edu.

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1 Introduction

The recent financial crises has brought the impact of leverage on financial system stability to the forefront. The crisis might well be understood as the bottom of a leverage cycle in which leverage and asset prices crashed together. It was preceded by years in which asset prices and the amount of leverage in the financial system increased dramatically. What determines leverage in equilibrium? Do these levels involve default? What is the effect of leverage and default on asset prices and the real side of the economy?

Our paper provides a complete characterization of leverage and default in *binomial* economies with *financial assets* serving as collateral.

Our first result, the Binomial No-Default Theorem, states that in binomial economies with financial assets serving as collateral, any equilibrium is equivalent (in real allocations and prices) to another equilibrium in which there is no default. Thus potential default has a dramatic effect on equilibrium, but actual default does not. The Binomial No-Default Theorem is valid in a very general context with arbitrary preferences and endowments, contingent and non-contingent promises, many assets, many consumption goods, multiple periods, and production.

The Binomial No-Default Theorem does not say that equilibrium is unique, only that each equilibrium can be replaced by another equivalent equilibrium in which there is no default. However, we show that among all equivalent equilibria, the equilibria which use the least amount of collateral never involve default. These collateral minimizing equilibria would naturally be selected if there were the slightest transactions cost in using collateral or handling default. In these equilibria we prove that the scale of promises per unit of collateral is unambiguously determined simply by the payoffs of the underlying collateral, independent of preferences or other fundamentals of the economy. Agents will promise as much as they can while assuring their lenders that the collateral is enough to guarantee delivery.

Our second result, the Binomial Leverage Theorem shows that when promises are non-contingent, as they typically are for the bulk of collateralized loans, the LTV on each financial asset in any collateral minimizing equilibrium is given by the following simple formula:

$$LTV = \frac{worst\,case\,rate\,of\,return}{riskless\,rate\,of\,interest}.$$

Equilibrium LTV for the family of non-contingent debt contracts is the ratio of the worst case return of the asset to the riskless rate of interest. Though simple and easy to calculate, this formula provides interesting insights. First, it explains which assets are easiest to leverage: the assets whose future value has the least bad downside can be leveraged the most. Second, it explains why changes in the bad tail can have such a big effect on equilibrium even if they hardly change expected payoffs: they change leverage. The theorem suggests that one reason leverage might have plummeted from 2006-2009 is because the worst case return that lenders imagined got much worse. Finally, the formula also explains why (even with rational agents who do not blindly chase yield), high leverage historically correlates with low interest rates.

The Binomial Leverage theorem shows that in static binomial models, leverage is endogenously determined in equilibrium by the Value at Risk equal zero rule, *assumed* by many other papers in the literature. We emphasize that LTV is the ratio of loan value to collateral value for each asset actually used as collateral. It may be that in some economies all the assets are used as collateral, while in other economies fewer assets are used as collateral. It is therefore very important to keep in mind another notion of leverage that we call *diluted* LTV, namely the ratio of total borrowing to total asset value (including identical assets not used as collateral). If nobody wants to borrow up to the debt capacity of his assets determined by our formula, then the collateral requirements are irrelevant, and debt is determined by the preferences of the agents in the economy, just as in models without collateral. In this case diluted LTV is smaller than LTV and we might say debt is determined by demand. On the other hand, if collateral is scarce, and agents are borrowing against all their collateralizable assets, then total borrowing is determined by the debt capacities of the assets, independently from agent preferences for borrowing. In this case we might say debt is determined by the supply of debt capacity. Thus when collateral is scarce, as it is in crises, leverage is exclusively controlled by shocks to anticipated asset returns.[1]

Our third result, the Binomial Leverage-Volatility Theorem, provides a precise link between leverage and volatility. When there are state "risk-neutral probabilities" such that all asset prices are equal to discounted expected payoffs, then the equilibrium margin $(1 - LTV)$ of an asset is proportional to the volatility of its returns.

[1] Our theory of endogenous leverage should be contrasted with the corporate finance theory of leverage we describe in the next section in which leverage rises or falls depending on the manager's incentives to steal the money.

This gives a rigorous foundation to the notion that leverage and margins are determined by volatility. When there is only one asset and one kind of loan, we show that risk neutral probabilities can always be found, despite the collateral constraints. But we hasten to add that in general, with many assets and loans, there will not be risk neutral probabilities that price all the assets in collateral equilibrium. Our LTV formula given in the Binomial Leverage Theorem nevertheless holds true in those cases as well. In binomial economies, it is therefore more accurate to say that leverage is determined by tail risk (as defined precisely by our formula) than it is to say that leverage is determined by volatility.

All our results depend on two key assumptions. First, we only consider financial assets, that is, assets that do not give direct utility to their holders, and which yield dividends that are independent of who holds them. Second, we assume that the economy is binomial, and that all loans are for one period.[2] Binomial economies are the simplest economies in which uncertainty can play an important role. It is not surprising therefore that they have played a central role in finance, such as in Black-Scholes pricing. A date-event tree in which loans last for just one period and every state is succeeded by exactly two nodes suggests a world with very short maturity loans and no big jumps in asset values, since Brownian motion can be approximated by binary trees with short intervals. Binomial models might thus be taken as good models of Repo markets, in which the assets do seem to be purely financial, and the loans are extremely short term, usually one day.

The No-Default Theorem implies that if we want to study consumption or production or asset price effects of actual default, we must do so in models that either include non-financial assets (like houses or asymmetrically productive land) or that depart from the standard binomial models used in finance. Our results also show that there is a tremendous difference between physical collateral that generates contemporaneous utility and backs long term promises, and financial collateral that gives utility only through dividends or other cash flows, and backs very short term promises. Our result might explain why there are some markets (like mortgages) in which defaults are to be expected while in others (like Repos) margins are set so strictly that default is almost ruled out.[3]

[2]We could also allow for a long term loan with one payment date, provided that all the states at that date could be partitioned into two events, on each of which the loan promise and the asset value is constant.

[3]Repo defaults, including of the Bear Stearns hedge funds, seem to have totaled a few billion dollars out of the trillions of dollars of repo loans during the period 2007-2009.

Finally, the No-Default Theorem has a sort of Modigliani-Miller feel to it. But the theorem does not assert that the debt-equity ratio is irrelevant. It shows that if we start from any equilibrium with default, we can move to an equivalent equilibrium, typically with less leverage, in which only no-default contracts are traded. The theorem does not say that starting from an equilibrium with no default, one can construct another equivalent equilibrium with even less leverage. Typically one cannot. Modigliani-Miller fails more generally in our model simply because any issuer of debt must put up collateral.

The paper is organized as follows. Section 2 presents the literature review. Section 3 presents a static model of endogenous leverage and debt with one asset and proves the main results in this simple case. Section 4 presents the general model of endogenous leverage and proves the general theorems. Section 5 presents two examples to illustrate our theoretical results.

2 Literature

To attack the leverage endogeneity problem we follow the techniques developed by Geanakoplos (1997). Agents have access to a menu of contracts, each of them characterized by a promise in future states and one unit of asset as collateral to back the promise. When an investor sells a contract she is borrowing money and putting up collateral; when she is buying a contract, she is lending money. In equilibrium every contract, as well as the asset used as collateral, will have a price. Each collateral-promise pair defines a contract, and every contract has a leverage or loan to value (the ratio of the price of the promise to the price of the collateral). The key is that even if all contracts are priced in equilibrium, because collateral is scarce, only a few will be actively traded. In this sense, leverage becomes endogenous. This earlier effort, however, did not give a practical recipe for computing equilibrium leverage. With our complete characterization, binomial models with financial assets become a completely tractable tool to study leverage and default.

Geanakoplos (2003, 2010), Fostel-Geanakoplos (2008, 2012a and 2012b), and Cao (2010), all work with binomial models of collateral equilibrium with financial assets, showing in their various special cases that, as the Binomial No-Default Theorem implies, only the *VaR= 0* contract is traded in equilibrium. These papers generally show that higher leverage leads to higher asset prices.

Other papers have already given examples in which the No-Default Theorem does not hold. Geanakoplos (1997) gave a binomial example with a non-financial asset (a house, from which agents derive utility), in which equilibrium leverage is high enough that there is default. Geanakoplos (2003) gave an example with a continuum of risk neutral investors with different priors and three states of nature in which the only contract traded in equilibrium involves default. Simsek (2013) gave an example with two types of investors and a continuum of states of nature with equilibrium default. Araujo, Kubler, and Schommer (2012) provided a two period example of an asset which is used as collateral in two different actively traded contracts when agents have utility over the asset. Fostel and Geanakoplos (2012a) provide an example with three periods and multiple contracts traded in equilibrium.

Many other papers have assumed a link between leverage and volatility (see for example Thurner et.al., 2012, and Adrian and Boyarchenko, 2012). There are two papers that derive this link from first principles. Fostel and Geanakoplos (2012a) show that an increase in volatility reduces leverage in a very special case of a binomial economy. In Brunnermeier and Sannikov (2013) leverage is endogenous but is determined not by collateral capacities but by agents risk aversion; it is a "demand-determined" leverage that would be the same without collateral requirements. The time series movements of LTV come there from movements in volatility because the added uncertainty makes borrowers more scared of investing, rather than from reducing the debt capacity of the collateral or making lenders more scared to lend.

This paper is related to a large and growing literature on collateral equilibrium and leverage. Some of these papers focus on investor-based leverage (the ratio of an agent's total asset value to his total wealth) as in Acharya and Viswanathan (2011), Adrian and Shin (2010), Brunnermeier and Sannikov (2013) and Gromb and Vayanos (2002). Other papers, such as Acharya, Gale and Yorulmazer (2011), Brunnermeier and Pedersen (2009), Cao (2010), Fostel and Geanakoplos (2008, 2012a and 2012b), Geanakoplos (1997, 2003 and 2010) and Simsek (2013), focus on asset-based leverage (as defined in this paper).

Not all these models actually make room for endogenous leverage. Often an ad-hoc behavioral rule is postulated. To mention just a few, Brunnermeier and Pedersen (2009) assume a VAR rule. Gromb and Vayanos (2002) and Vayanos and Wang (2012) assume a max min rule that prevents default. Some other papers like Garleanu and Pedersen (2011) and Mendoza (2010) assumed a fixed LTV.

In other papers leverage is endogenous, though the modeling strategy is not as in our paper. In the corporate finance approach of Bernanke and Gertler (1989), Kiyotaki and Moore (1997), Holmstrom and Tirole (1997), Acharya and Viswanathan (2011) and Adrian and Shin (2010) the endogeneity of leverage relies on asymmetric information and moral hazard problems between lenders and borrowers. Asymmetric information is important in loan markets for which the borrower is also a manager who exercises control over the value of the collateral. Lenders may insist that the manager puts up a portion of the investment himself in order to maintain his skin in the game. The recent crisis, however, was centered not in the corporate bond world, where managerial control is central, but in the mortgage securities market, where the buyer/borrower generally has no control or specialized knowledge over the cash flows of the collateral.

3 Leverage and Default in a Simple Model of Debt.

We first restrict our attention to a simple static model with only two periods, one asset, and non-contingent debt contracts and prove our results in this framework.

3.1 Model

3.1.1 Time and Assets

We begin with a simple two-period general equilibrium model, with time $t = 0, 1$. Uncertainty is represented by different states of nature $s \in S$ including a root $s = 0$. We denote the time of s by $t(s)$, so $t(0) = 0$ and $t(s) = 1, \forall s \in S_T$, the set of terminal nodes of S. Suppose there is a single perishable consumption good c and one asset Y which pays dividends d_s of the consumption good in each final state $s \in S_T$. We take the consumption good as numeraire and denote the price of Y at time 0 by p.

We call the asset a *financial asset* because it gives no direct utility to investors, and pays the same dividends no matter who owns it. Financial assets are valued exclusively because they pay dividends. Houses are not financial assets because they give utility to their owners. Neither is land if its output depends on who owns it and tills it.

3.1.2 Investors

Each investor $h \in H$ is characterized by a utility, u^h, a discount factor, δ_h, and subjective probabilities, γ_s^h, $s \in S_T$. We assume that the utility function for consumption in each state $s \in S$, $u^h : R_+ \to R$, is differentiable, concave, and monotonic. The expected utility to agent h is:[4]

$$U^h = u^h(c_0) + \delta_h \sum_{s \in S_T} \gamma_s^h u^h(c_s). \tag{1}$$

Investor h's endowment of the consumption good is denoted by $e_s^h \in R_+$ in each state $s \in S$. His endowment of the only asset Y at time 0 is $y_{0*}^h \in R_+$. We assume that the consumption good is present in every state, $\sum_{h \in H} e_0^h > 0, \sum_{h \in H}(e_s^h + d_s y_{0*}^h) > 0, \forall s \in S_T$.

3.1.3 Collateral and Debt.

A *debt contract* j promises $j > 0$ units of consumption good in each final state backed by one unit of asset Y serving as collateral. The terms of the contract are summarized by the ordered pair $(j \cdot \tilde{1}, 1)$. The first component, $j \cdot \tilde{1} \in R^{S_T}$ (the vector of j's with dimension equal to the number of final states) denotes the (non-contingent) promise. The second component, 1, denotes the one unit of the asset Y used as collateral. Let J be the set of all such available debt contracts.

The price of contract j is π_j. An investor can borrow π_j today by selling the debt contract j in exchange for a promise of j tomorrow. Let φ_j be the number of contracts j traded at time 0. There is no sign constraint on φ_j; a positive (negative) φ_j indicates the agent is selling (buying) $|\varphi_j|$ contracts j or borrowing (lending) $|\varphi_j|\pi_j$.

We assume the loan is non-recourse, so the maximum a borrower can lose is his collateral if he does not honor his promise: the actual delivery of debt contract j in state $s \in S_T$ is $min\{j, d_s\}$. If the promise is small enough that $j \leq d_s, \forall s \in S_T$, then the contract will not default. In this case its price defines a riskless rate of interest $(1 + r_j) = j/\pi_j$.

[4] All that matters for the results in this paper is that the utility $U^h : \mathbb{R}^{1+S} \to \mathbb{R}$ depends only on consumption (and not on portfolio holdings). The expected utility representation is done for familiarity. Our results will not depend on any specific type of agent heterogeneity either.

The *Loan to Value* (LTV) associated to debt contract j is given by

$$LTV_j = \frac{\pi_j}{p}. \tag{2}$$

The margin requirement m_j associated to debt contract j is $1 - LTV_j$, and the leverage associated to debt contract j is the inverse of the margin, $1/m_j$.

We define the *average loan to value*, LTV for asset Y, as the trade-value weighted average of LTV_j across all debt contracts actively traded in equilibrium, and the *diluted average loan to value*, LTV_0^Y (which includes assets with no leverage) by

$$LTV^Y = \frac{\sum_h \sum_j \max(0, \varphi_j^h)\pi_j}{\sum_h \sum_j \max(0, \varphi_j^h)p} \geq \frac{\sum_h \sum_j \max(0, \varphi_j^h)\pi_j}{\sum_h y_{0^*}^h p} = LTV_0^Y.$$

3.1.4 Budget Set

Given the asset and debt contract prices $(p, (\pi_j)_{j \in J})$, each agent $h \in H$ decides consumption, c_0, asset holding, y, and debt contract trades, φ_j, at time 0, and also consumption, c_s, in each state $s \in S_T$, in order to maximize utility (1) subject to the budget set defined by

$$B^h(p, \pi) = \{(c, y, \varphi) \in R_+^S \times R_+ \times R^J :$$

$$(c_0 - e_0^h) + p(y - y_{0^*}^h) \leq \textstyle\sum_{j \in J} \varphi_j \pi_j$$

$$(c_s - e_s^h) \leq y d_s - \textstyle\sum_{j \in J} \varphi_j min(j, d_s), \forall s \in S_T$$

$$\textstyle\sum_{j \in J} max(0, \varphi_j) \leq y\}.$$

At time 0, expenditures on consumption and the asset, net of endowments, must be financed by money borrowed using the asset as collateral. In the final period, at each state s, consumption net of endowments can be at most equal to the dividend payment minus debt repayment. Finally, those agents who borrow must hold the required collateral at time 0. Notice that even with as many independent contracts as there are terminal states, equilibrium might still be different from Arrow-Debreu. Agents cannot willy nilly combine these contracts to sell Arrow securities because they need to post collateral.[5]

3.1.5 Collateral Equilibrium

A *Collateral Equilibrium* is a set consisting of an asset price, debt contract prices, individual consumptions, asset holdings, and contract trades $((p, \pi), (c^h, y^h, \varphi^h)_{h \in H}) \in (R_+ \times R_+^J) \times (R_+^S \times R_+ \times R^J)^H$ such that

1. $\sum_{h \in H}(c_0^h - e_0^h) = 0$.
2. $\sum_{h \in H}(c_s^h - e_s^h) = \sum_{h \in H} y^h d_s, \forall s \in S_T$.
3. $\sum_{h \in H}(y^h - y_{0*}^h) = 0$.
4. $\sum_{h \in H} \varphi_j^h = 0, \forall j \in J$.
5. $(c^h, y^h, \varphi_j^h) \in B^h(p, \pi), \forall h$
   $(c, y, \varphi) \in B^h(p, \pi) \Rightarrow U^h(c) \leq U^h(c^h), \forall h$.

Markets for the consumption good in all states clear, assets and promises clear in equilibrium at time 0, and agents optimize their utility in their budget sets. As shown by Geanakoplos and Zame (1997), equilibrium in this model always exists under the assumption we have made so far.[6]

3.2 The Binomial No-Default Theorem

3.2.1 The Theorem

Consider the situation in which there are only two terminal states, $S = \{0, U, D\}$. Asset Y pays d_U units of the consumption good in state $s = U$ and $0 < d_D < d_U$

[5]Notice that we are assuming that short selling of assets is not possible. This is part of the assumption that agents need to post collateral in order to sell future promises. In this new framework, short sales are explicitely modeled using the collateral terminology. In Fostel-Geanakoplos (2012b) we investigate the significance of short selling and CDS for asset pricing.

[6]The set H of agents can be taken as finite (in which case we really have in mind a continuum of agents of each of the types), or we might think of $H = [0, 1]$ as a continuum of distinct agents, in which case we must think of all the agent characteristics as measurable functions of h. In the latter case we must think of the summation $\sum$ over agents as an integral over agents, and all the optimization conditions as holding with Lebesgue measure one.

in state $s = D$.[7] Figure 1 depicts the asset payoff. Default occurs in equilibrium if and only if some contract j with $j > d_D$ is traded. One might imagine that some agents value the asset much more than others, say because they attach very high probability γ_U^h to the U state, or because they are more risk tolerant, or because they have very low endowments e_U^h in the U state, or because they put a high value δ^h on the future. These agents might be expected to want to borrow a lot, promising $j > d_D$ so as to get their hands on more money to buy more assets at time 0. Indeed it is true that for $j > j^* = d_D$, any agent can raise more money $\pi_j > \pi_{j_*}$ by selling contract j rather than j^*. Nonetheless, as the following result shows, we can assume without loss of generality that the only debt contract traded in equilibrium will be the max min contract j^*, on which there is no default.

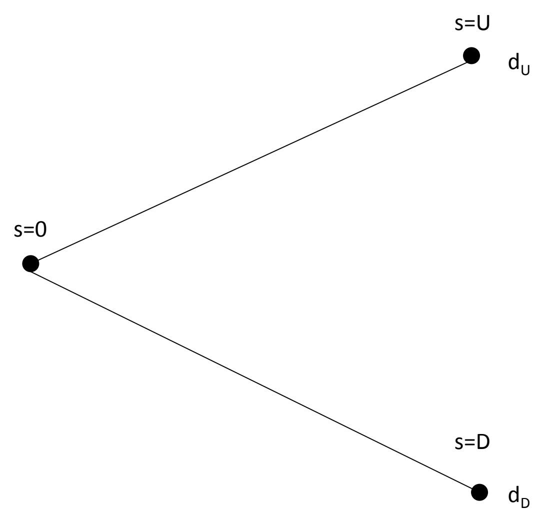

Fig. 1: Asset payoff description.

[7]Without loss of generality, $d_U \geq d_D$. If $d_D = 0$ or $d_D = d_U$, then the contracts are perfect substitutes for the asset, so there is no point in trading them. Sellers of the contracts could simply hold less of the asset and reduce their borrowing to zero while buyers of the contracts could buy the asset instead. So we might as well assume $0 < d_D < d_U$.

Binomial No-Default Theorem:

Suppose that $S = \{0, U, D\}$, that Y is a financial asset, and that the max min debt contract $j^* = d_D \in J$. Then given any equilibrium $((p, \pi), (c^h, y^h, \varphi^h)_{h \in H})$, we can construct another equilibrium $((p, \pi), (c^h, \bar{y}^h, \bar{\varphi}^h)_{h \in H})$ with the same asset and contract prices and the same consumptions, in which j^* is the only debt contract traded, $\bar{\varphi}^h_j = 0$ if $j \neq j^*$. Hence equilibrium default can be taken to be zero.

Proof:

The proof is organized in three steps.

1. Payoff Cone Lemma.

The portfolio of assets and contracts that any agent h holds in equilibrium delivers payoff vector (w^h_U, w^h_D) which lies in the cone positively spanned by $(d_U - j^*, 0)$ and *(j^*, j^*). The U Arrow security payoff $(d_U - j^*, 0) = (d_U, d_D) - (j^*, j^*)$ can be obtained from buying the asset while simultaneously selling the max min debt contract.*

Any portfolio payoff (w_U, w_D) is the sum of payoffs from individual holdings. The possible holdings include debt contracts $j > j^*, j = j^*, j < j^*$, the asset, and the asset bought on margin by selling some debt contract j. The debt contracts and the asset all deliver at least as much in state U as in state D. So does the leveraged purchase of the asset. In fact, buying the asset on margin using any debt contract with $d_U > j \geq j^*$ is effectively a way of buying the U Arrow payoff $(d_U - j, 0)$. This can be seen in Figure 2.

In short, we have that the Arrow U security and the max min debt contract positively span all the feasible payoff space, as shown in Figure 3.

2. State Pricing Lemma.

There exist unique state prices $a > 0, b > 0$ such that if any agent h holds a portfolio delivering (w^h_U, w^h_D), the portfolio costs $aw^h_U + bw^h_D$.

In steps (a) and (b) we find state prices for two securities: the asset and the max min debt contract j^*. In steps (c) and (d) we use the Payoff Cone Lemma to show that the same state prices can be used to price any other debt contract $j \neq j^*$ that is traded in equilibrium. The cost of any portfolio is obtained as the sum of the costs of its constituent parts.

(a) There exist unique a and b pricing the asset and the max min contract, that is solving $\pi_{j*} = aj^* + bj^*$ and $p = ad_U + bd_D$.

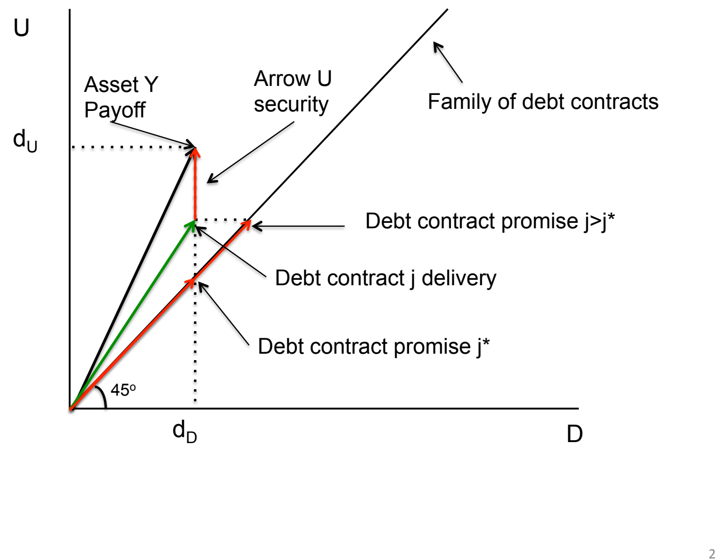

Fig. 2: Creating the U Arrow security.

Since $d_U > d_D$ the two equations are linearly independent and therefore there exists a unique solution (a, b). It is easy to check that $a = \frac{p-\pi_{j*}}{d_U - j^*}$ *and* $b = \pi_{j*}/j^* - a$.

Notice that $a(d_U - j^*)$ is the price of buying $(d_U - j^*)$ Arrow U securities obtained by buying the asset Y and selling the max min contract j^*.

(b) State prices are positive, that is, $a > 0$ *and* $b > 0$.

First, $a > 0$, otherwise agents could buy more of Arrow U at a lower cost, violating agent optimization in equilibrium. We must also have $b > 0$, otherwise nobody would hold the asset Y in equilibrium since it would be better to buy d_U/j^* units of the contract j^* which delivers the same d_U in the up state and more $d_U > d_D$ in the down state and costs at most the same, namely $p + b(d_U - d_D) \leq p$.

(c) Suppose debt contract j *with* $j \neq j^* = d_D$ *is positively traded in equilibrium. Then* $\pi_j \leq a \cdot min\{d_U, j\} + b \cdot min\{j^*, j\}$.

By the Positive Cone Lemma, the delivery of contract j is positively spanned by the Arrow U security $(d_U - j^*, 0)$ and the max min contract (j^*, j^*), both of which are priced by a and b. Hence any buyer could obtain the same deliveries buy buying a positive linear combination of the two, which would then be priced by a and b. This

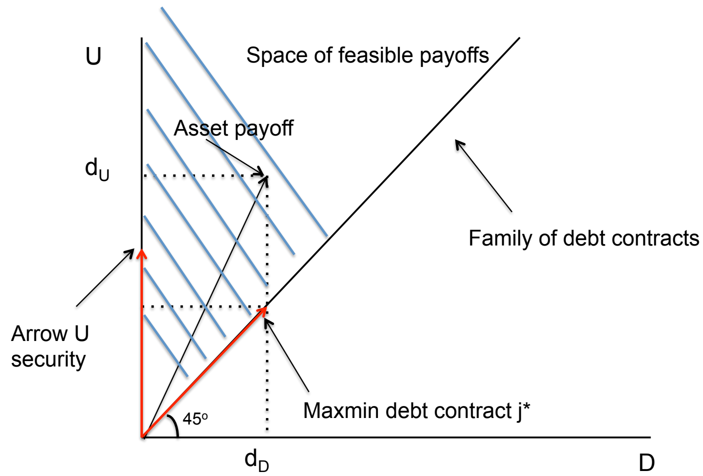

Fig. 3: Positive spanning.

provides the upper-bound for π_j.

(d) Suppose debt contract j with $j \neq j^* = d_D$ is positively traded in equilibrium. Then $\pi_j \geq a \cdot min\{d_U, j\} + b \cdot min\{j^*, j\}$.

In case $j \leq j^* = d_D$, the contract fully delivers j in both states, proportionally to contract j^*. If its price were less than $\pi_{j^*}(j/j^*) = aj + bj$, its sellers should have sold j/j^* units of j^* instead, which would have been feasible for them as it requires less collateral.

Consider the case $j > j^*$. Any seller of contract j has entered into a double trade, buying (or holding) the asset as collateral and selling contract j, at a net cost of $p - \pi_j$. Since any contract $j > d_U$ delivers exactly the same way in both states as contract $j = d_U$, we can without loss of generality restrict attention to contracts j with $d_D < j \leq d_U$. Any agent selling such a contract, while holding the required collateral, receives on net $d_U - j$ in state U, and nothing in state D. The key is that the seller is actually a buyer of the Arrow U. The cost is $p - \pi_j$ which, given step (a), is at most $a(d_U - j)$. Hence

$$p - \pi_j \leq a(d_U - j)$$

$$\pi_j \geq p - a(d_U - j)$$

$$\pi_j \geq ad_U + bd_D - a(d_U - j)$$

$$\pi_j \geq aj + bj^* = a \cdot min\{d_U, j\} + b \cdot min\{j^*, j\}.$$

3. Construction of the new default-free equilibrium

Define

$$(w_U^h, w_D^h) = y^h(d_U, d_D) - \sum_j (\min(j, d_U), \min(j, d_D))\varphi_j^h.$$

$$\bar{y}^h = \frac{w_U^h - w_D^h}{d_U - d_D}.$$

$$\bar{\varphi}_{j^*}^h = [\bar{y}^h d_D - w_D^h]/j^* = \bar{y}^h - w_D^h/j^*.$$

If in the original equilibrium, y^h is replaced by $\bar{y}^h$ and φ_j^h is replaced by 0 for $j \neq j^*$ and by $\bar{\varphi}_{j^*}^h$ for $j = j^*$, and all prices and other individual choices are left the same, then we still have an equilibrium.

(a) Agents are maximizing in the new equilibrium.

Note that $\bar{\varphi}_{j^*}^h \leq \bar{y}^h$, so this portfolio choice satisfies the collateral constraint.

Using the above definitions, the net payoff in state D is the same as in the original equilibrium,

$$\bar{y}^h d_D - \bar{\varphi}_{j^*}^h j^* = w_D^h$$

and the same is also true for the net payoff in state U,

$$\bar{y}^h d_U - \bar{\varphi}_{j^*}^h j^* = \bar{y}^h(d_U - d_D) + w_D^h = (w_U^h - w_D^h) + w_D^h = w_U^h.$$

Hence the portfolio choice $(\bar{y}^h, \bar{\varphi}_{j^*}^h)$ gives the same payoff (w_U^h, w_D^h). From the previous Lemmas, the newly constructed portfolio must have the same cost as well. Since Y is a financial asset, every agent is optimizing.

(b) Markets clear in the new equilibrium.

Summing over individuals we must get

$$\sum_h \bar{y}^h(d_U, d_D) - \sum_h \bar{\varphi}_{j^*}^h(j^*, j^*) =$$

$$\sum_h (w_U^h, w_D^h) = \sum_h y^h(d_U, d_D) - \sum_h \sum_j \varphi_j^h(\min(j, d_U), \min(j, d_D)) = \sum_h y^h(d_U, d_D).$$

The first equality follows from step (a), the second from the definition of net payoffs in the original equilibrium, and the last equality follows from the fact that $\sum_h \varphi_j^h = 0$ in the original equilibrium for each contract j. Hence we have that

$$\sum_h (\bar{y}^h - y^h)(d_U, d_D) - \sum_h \bar{\varphi}_{j^*}^h(j^*, j^*) = 0.$$

By the linear independence of the vectors (d_U, d_D) and (j^*, j^*) we deduce that

$$\sum_h \bar{y}^h = \sum_h y^h$$
$$\sum_h \bar{\varphi}_{j^*}^h = 0.$$

Hence markets clear.■

We now call attention to an interesting corollary of the proof just given. By modifying the equilibrium prices in the above construction for contracts that are not traded, we can bring them into line with the state prices a, b defined in the proof of the Binomial No-Default Theorem, without affecting equilibrium. More concretely,

Binomial State Pricing Corollary:

Under the conditions of the Binomial No-Default Theorem we may suppose that the new no-default equilibrium has the property that there exist unique state prices $a > 0$ and $b > 0$, such that $p = ad_U + bd_D$, and $\pi_j = a \cdot \min\{d_U, j\} + b \cdot \min\{d_D, j\}$, $\forall j \in J$.

Proof:

The proof was nearly given in the proof of the Binomial No-Default Theorem. It is straightforward to show that if a previously untraded contract has its price adjusted into line with the state prices, then nothing is affected.■

3.2.2 Discussion

The Binomial No-Default Theorem shows that in any static binomial model with a single financial asset, we can assume without loss of generality that the *only* debt contract actively traded is the max min debt contract, on which there is no default.

Thus *potential* default has a dramatic effect on equilibrium, but *actual* default does not. In other words, default does not alter the span of contract payoffs.

The Binomial No-Default Theorem does not say that equilibrium is unique, only that each equilibrium can be replaced by another with the *same asset price and the same consumption* by each agent, in which there is no default.

The Binomial No-Default Theorem has a sort of Modigliani-Miller feel to it. But the theorem does *not* assert that the debt-equity ratio is irrelevant. The theorem shows that if we start from any equilibrium, we can move to an equivalent equilibrium in which only max min debt is traded. If the original equilibrium had default, in the new equilibrium, leverage will be lower. Thus starting from a situation of default, the theorem does say that leverage can be lowered over a range until the point of no default, while leaving all investors indifferent. The theorem does *not* say that starting from a max min equilibrium, one can construct another equilibrium with still lower leverage, or even with higher leverage. Modigliani-Miller does not fully hold in our model because issuers of debt must hold collateral. In the traditional proof of Modigliani-Miller, when the firm issues less debt, the buyer of its equity compensates by issuing debt himself. But this arguments relies on the fact that the equity holder has enough collateral. In our model, if less debt is issued on a unit of collateral, then that collateral is wasted and there may not be enough other free collateral to back the new debt. In Section 5 we give an example with a unique equilibrium in which every borrower leverages to the max min, but no agent would be indifferent to leveraging any less. In that example Modigliani-Miller completely fails.

In the Binomial State Pricing Corollary, the state prices a, b are like Arrow prices. Their existence implies that there are no arbitrage possibilities in trading the asset and the contracts. Even a trader who had infinite wealth and who was allowed to make promises without putting up any required collateral (but delivered as if he put up the collateral) could not find a trade that made money in some state without ever losing money. However, the equilibrium may not be an Arrow-Debreu equilibrium, even though the state prices are uniquely defined. We shall see an example with unique state prices but Pareto inferior consumptions (coming from the collateral constraints) in Section 5.

Finally, let us provide some intuition to the proof of the No-Default Therem. There are two key assumptions. First, we only consider financial assets, that is, assets that do not give direct utility at time 0 to their holders, and which yield dividends at time 1 that are independent from who holds them at time 0. Second, we assume that the tree is binary.

In the first step of the proof, the Payoff Cone Lemma shows that the max min promise plus the Arrow U security (obtained by buying the asset while selling the max min debt contract), positively spans the cone of all feasible portfolio payoffs. The assumption of two states is crucial. If there were three states, it might be impossible for a portfolio holder to reproduce his original net payoffs from a portfolio in which he can only hold the asset and buy or issue the max min debt.

In the second step of the proof, the State Pricing lemma shows that any two portfolios that give the same payoffs in the two states must cost the same. One interesting feature of the proof is that it demonstrates the existence of state prices (that price the asset and all the debt contracts) even though short-selling is not allowed. In general, if an instrument (asset or bond) C has payoffs that are a positive combination of the payoffs from instruments A and B, then the price of C cannot be above the positive combination of the prices of A and B. Otherwise, any buyer or holder of C could improve by instead combining the purchase of A and B. This logic gives an *upper-bound* for prices of all traded instruments. On the other hand, the price of C could be less than the price of the positive combination of A and B (and just slightly more than the individual prices of A and B) because there may be no agent interested in buying it, and the sellers cannot split C into A and B. Nonetheless, we show that we can also get a *lower-bound* for the price of C. The reason is that in our model, the sellers of the debt contract must own the collateral, and hence on *net* are in fact buyers of something that lies in the positive cone, which gives us an upper bound for the price of what they buy, and hence the missing lower bound on what they sell. In short, the crucial argument in the proof is that sellers of contracts are actually buyers of something else that is in the payoff cone. As we will see later, when there are multiple assets, or multiple kinds of loans on the same asset, the sellers of a bond in one family may not be purchasing something in the payoff cone of another family. Each family may require different state prices. That is why the No-Default Theorem holds more generally, but the State Pricing Corollary does not.

In the third step of the proof we use both lemmas to show that in equilibrium each agent is indifferent to replacing his portfolio with another such that on each unit of collateral that he holds, he either leverages to the maximum amount without risk of default, or does not leverage at all. The idea is as follows. If in the original equilibrium the investor leveraged his asset purchases *less* than the max min, he could always leverage some of his holdings to the max min, and the others not at all. This of course *reduces* the amount of asset he uses for collateral. If in the original equilibrium the investor was selling *more* debt than the max min, defaulting in the D state, then he could again *reduce* his asset holdings and his debt sales to the max min level per unit of asset held, and still end up buying the same amount of the Arrow U security.[8] The reason he can reproduce his original net payoffs despite issuing fewer bonds per unit of collateral is that, on net, all contracts $j > j^*$ leave the collateral holder with some multiple of the Arrow U security. He simply must compensate by leveraging a different amount of collateral. By selling less debt per unit of collateral, he must spend more cash on each unit of the asset, so the reduction in asset holdings should not be surprising.[9] Once we see that the debt issuer can maintain the same net payoffs even if he issues less debt per unit of collateral, it is easy to see that his new behavior can be made part of a new equilibrium. Let the original buyer of the original risky bond buy instead all of the new max min debt plus all the asset that the original risky bond seller no longer holds. By construction the total holdings of the asset is unchanged, and the total holdings of debt is zero, as before. Furthermore, by construction, the seller of the bond has the same portfolio payoff as before, so he is still optimizing. Since the total payoff is just equal to the dividends from the asset, and that is unchanged, the buyer of the bond must also end up with the same payoffs in the two states, so he is optimizing as well. The new portfolio may involve each agent holding a new amount of the collateral asset, while getting the same payoff from his new portfolio of assets and contracts. Agents are indifferent to switching to the new portfolio because of the crucial assumption that the asset is a financial asset. If the collateral were housing or productive land for example, the theorem would not hold.

3.3 Equilibrium Refinements

The Binomial No-Default Theorem states that every collateral equilibrium is equivalent to a j^* equilibrium in which there is no default and j^* is the only contract

[8]If he continued to hold the same assets while reducing his debt to the max min per asset, then he would end up with more of the Arrow U security.

[9]To put it in other words, the debt on which he was defaulting provided deliveries that were similar to the asset (more in the up state than in the down state), so when he sells less of these he must compensate by selling more of the asset and thus holding less.

traded. But the proof reveals more, namely that in the j^* equilibrium every agent uses less of the asset as collateral, with one agent using strictly less, than in any other equivalent equilibrium. Thus, our theorem can be further sharpened if we add to the model some cost structure associated to either default or collateral use. More precisely, the following results hold.

No-Default Theorem Refinement 1: Default Costs.

Suppose that $\epsilon > 0$ units of the consumption good are lost after default. Then in every equilibrium only debt contracts $j \leq j^*$ will be traded.

Proof:

The proof follows immediately from the portfolio construction procedure in the Binomial No-Default Theorem, since for all $j > j^*$ agents will incur unnecessary default costs.■

The last theorem shows that if we add to the model a small cost to default, then our No-Default theorem has more bite: now the equilibrium prediction always rules out default. Notice, however, that the equilibrium contracts may not be unique, in the sense that agents may be leveraging less than in the max min level. The following results sharpens our theorem even more.

No-Default Theorem Refinement 2: Collateral Costs.

Suppose that $\epsilon > 0$ units of the consumption good are lost for every unit of asset used as collateral. Then in every equilibrium only the debt contract j^* will be traded.

Proof:

The proof follows immediately from the portfolio construction procedure in the Binomial No-Default Theorem, since it is always the case that if $j \neq j^*$ is traded in equilibrium, then some agent is using more collateral than would be required if he only sold j^*.■

The last refinement shows that if we add to the model a small cost associated to collateral use, then j^* is the only contract traded in any equilibrium. This extra assumption is arguably realistic: examples of such costs are lawyer fees, intermediations fees, or even the more recent services provided by banks in the form of collateral transformation.

3.4 Binomial Leverage Theorem

3.4.1 The theorem

The previous theorem gives an explicit formula for how many promises every unit of collateral will back in equilibrium, or equivalently, how much collateral will be needed to back each promise of one unit of consumption in the future. Leverage is usually defined in terms of a ratio of value to value, which also admits a simple formula. We now provide a characterization of endogenous leverage.

Binomial Leverage Theorem:

Suppose that $S = \{0, U, D\}$, that Y is a financial asset, and that the max min debt contract $j^* = d_D \in J$. Then equilibrium LTV^Y can be taken equal to $\frac{\pi_j^*}{p} = \frac{d_D/(1+r_{j^*})}{p} = \frac{d_D/p}{1+r_{j^*}} = \frac{worst\ case\ rate\ of\ return}{riskless\ rate\ of\ interest}$.

Proof:

The proof follows directly from the Binomial No-Default Theorem. Since we can assume that in equilibrium the only contract traded is j^*, then

$$\frac{\pi_j^*}{p} = \frac{d_D/(1+r_{j^*})}{p} = \frac{d_D/p}{1+r_{j^*}}. \blacksquare$$

3.4.2 Discussion

The Binomial Leverage Theorem provides a very simple prediction about equilibrium leverage. According to the theorem, equilibrium LTV^Y for the family of non-contingent debt contracts is the ratio of the worst case return of the asset divided by the riskless rate of interest.

Equilibrium leverage depends on current and future asset prices, but is otherwise independent of the utilities or the endowments of the agents. The theorem shows that in static binomial models, leverage is endogenously determined in equilibrium by the Value at Risk equal zero rule, assumed by many other papers in the literature.

Though simple and easy to calculate, this formula provides interesting insights. First, it explains why changes in the bad tail can have such a big effect on equilibrium even if they hardly change expected payoffs: they change leverage. The theorem suggests that one reason leverage might have plummeted from 2006-2009 is because the worst case return that lenders imagined got much worse. Second, the formula also explains why (even with rational agents who do not blindly chase yield), high leverage historically correlates with low interest rates. Finally, it explains which assets are easier to leverage. In Section 4 we shall allow for leverage on multiple assets. The same formula for each asset shows that, given equal prices, the asset whose future value has the least bad downside can be leveraged the most.

Collateralized loans always fall into two categories. In the first category, a borrower is not designating all the assets he holds as collateral for his loans. In this case he would not want to borrow any more at the going interest rates even if he did not need to put up collateral (but was still required, by threat of punishment, to deliver the same payoffs he would had he put up the collateral). His demand for loans is then explained by conventional textbook considerations of risk and return. If all borrowers are in this case, then the rate of interest clears the loan market without consideration of collateral or default. In the second category, some or all the borrowers might be posting all their assets as collateral. In this case of scarce collateral, the loan market clears at a level determined by the spectre of default.

In short, there are two regimes. First, when all the assets fall into the first category, we can say that the debt in the economy is determined by the demand for loans. When all the assets and borrowers fall into the second category, we can say the debt in the economy is determined by the supply of credit, that is, by the maximum debt capacity of the assets. In binomial models with financial assets, the equilibrium LTV^Y can be taken to be the same easy to compute number, no matter which category the loan is in, that is whether it is demand or supply determined.

The distinction between plentiful and scarce capital all supporting loans at the same LTV^Y suggests that it is useful to keep track of a second kind of leverage that we introduced in Section 3.1.3 and called *diluted* leverage, LTV_0^Y. Consider the following example: if the asset is worth \$100 and its worst case payoff determines a debt capacity of \$80, then in equilibrium we can assume all debt loans written against this asset will have LTV^Y equal to 80%. If an agent who owns the asset only wants to borrow \$30, then she could just as well put up only three eights of the asset as collateral, since that would ensure there would be no default. The LTV^Y would then again be \$30/\$37.50 or 80%. Hence, it is useful to consider diluted LTV_0^Y, namely the ratio of the loan amount to the total value of the asset, even if some of the asset is not used as collateral. The diluted LTV_0^Y in this example is 30%, because the denominator includes the \$62.50 of asset that was not used as collateral. Finally, it is worth noting that in the proof of the Binomial No-Default Theorem in moving from an old equilibrium in which only contracts $j < j^*$ are traded to the new max min equilibrium, diluted leverage stays the same, but leverage on the margined assets rises. In moving from an old equilibrium with default in which a contract $j > j^*$ is traded to the new max min equilibrium, diluted leverage strictly declines, and leverage on the margined assets also declines.

3.5 Binomial Leverage-Volatility Theorem

It is often said that leverage should be related to volatility. The next theorem allow us to prove the connection, provided we measure volatility with respect to probabilities (the so-called risk neutral probabilities) $\alpha = (1{+}r)a$, $\beta = (1{+}r)b$ defined by the state prices a, b. Define the expectation and volatility of the asset payoffs with respect to α, β by

$$E_{\alpha,\beta}(Y) = d = \alpha d_U + \beta d_D$$

$$Vol_{\alpha,\beta}(Y) = \sqrt{\alpha(d_U - d)^2 + \beta(d_D - d)^2} = \sqrt{\alpha\beta}(d_U - d_D)$$

The following theorem shows that the margin on a loan using Y as collateral can be taken to be proportional to the volatility of its returns.

Binomial Leverage-Volatility Theorem:

Suppose that $S = \{0, U, D\}$, that Y is a financial asset, and that the max min debt contract $j^* = d_D \in J$. Then equilibrium margin m = 1-LTV can be taken equal to

$$m = \sqrt{\frac{\alpha}{\beta}} \frac{Vol_{\alpha,\beta}(Y)}{(1 + r_{j^*})p}.$$

Proof:

$$LTV = \frac{\pi_j^*}{p} = \frac{d_D}{(1 + r_{j^*})p}$$

$$m = 1 - LTV = \frac{(1 + r_{j^*})(ad_U + bd_D)}{(1 + r_{j^*})p} - \frac{d_D}{(1 + r_{j^*})p}$$

$$= \frac{\alpha d_U + \beta d_D}{(1 + r_{j^*})p} - \frac{d_D}{(1 + r_{j^*})p} = \frac{\alpha(d_U - d_D)}{(1 + r_{j^*})p} = \sqrt{\frac{\alpha}{\beta}} \frac{Vol_{\alpha,\beta}(Y)}{(1 + r_{j^*})p}. \blacksquare$$

The trouble with this theorem is that the risk neutral probabilities are not invariant across economies. If the asset payoffs d_U, d_D were to change, the risk neutral probabilities would change also. We could not unambiguously say leverage went down because volatility went up. And as we shall see in Section 4, if there were two assets Y and Z co-existing in the same economy, we might need different risk neutral probabilities to price Y and its debt than we would to price Z and its debt. Ranking the leverage of assets by the volatility of their payoffs would fail if we tried to measure the various volatilities with respect to the same probabilities.

4 A General Binomial Model.

In this section we show that the irrelevance of actual default is a much more general phenomenon, as long as we maintain our two key assumptions: financial assets and binary payoffs. We allow for the following extensions.

Arbitrary one-period contracts: previously we assumed that the only possible contract promise was non-contingent debt. Now we allow for arbitrary promises (j_U, j_D), provided that the max min version of the promise $(\bar{\lambda} j_U, \bar{\lambda} j_D)$ where $\bar{\lambda} = \max\{\lambda \in R_+ : \lambda(j_U, j_D) \leq (d_U, d_D)\}$ is also available.

Multiple simultaneous kinds of one-period contracts: not only can the promises be contingent, there can also be many different (non-colinear) types of promises co-existing. See Figure 4.

Multiple assets: we can allow for many different kinds of collateral at the same time, each one backing many (possibly) non-colinear promises.

Production and degrees of durability: the model already implicitly includes the storage technology for the asset. Now we allow the consumption goods to be durable, though their durability may be imperfect. We also allow for intra-period production. In fact, we allow for general production sets, provided that the collateral stays sequestered, and prevented from being used as an input.

Multiple goods: unlike our previous model, in each state of nature there will be more than one consumption good.

Multiple periods: we will extend our model to a dynamic model with an arbitrarily (finite) number of periods, as long as the tree is binomial.

Multiple states of nature: in each point in time, we will allow for multiple branches, as long as each (asset payoff, contract promise) pair takes on at most two values.

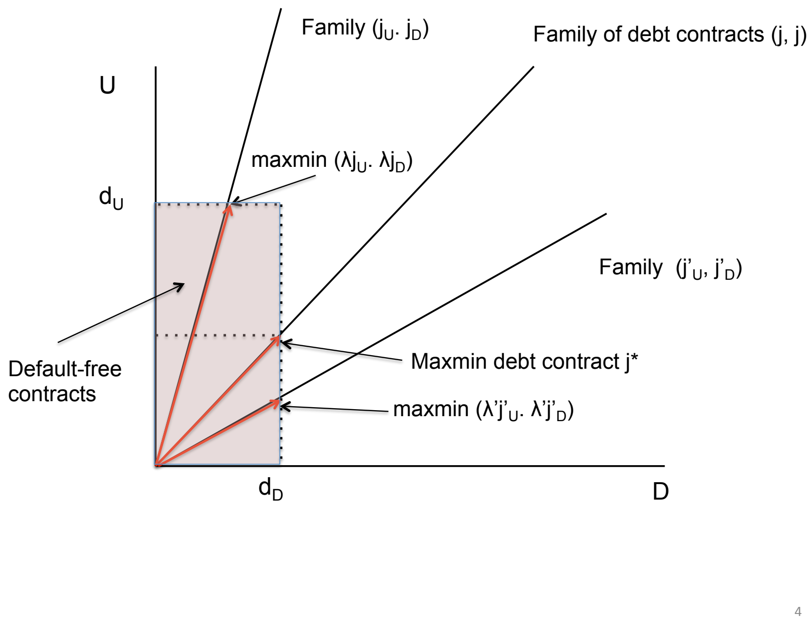

Fig. 4: Different types of contingent contracts

4.1 Model

4.1.1 Time and Assets

Uncertainty is represented by the existence of different states of nature in a finite tree $s \in S$ including a root $s = 0$, and terminal nodes $s \in S_T$. We denote the time of s as $t(s)$, so $t(0) = 0$. Each state $s \neq 0$ has a unique immediate predecessor s^*, and each non-terminal node $s \in S \backslash S_T$ has a set $S(s)$ of immediate successors.

Suppose there are $L = \{1, ..., L\}$ consumption goods ℓ and $K = \{1, ..., K\}$ financial assets k which pay dividends $d_s^k \in \mathbb{R}_+^L$ of the consumption goods in each state $s \in S$. The dividends d_s^k are distributed at state s to the investors who owned the asset in state s^*.

Finally, $q_s \in R_+^L$ denotes the vector of consumption goods prices in state s, whereas $p_s \in R_+^K$ denotes the asset prices in state s.

4.1.2 Investors

Each investor $h \in H$ is characterized by a utility, u^h, a discount factor, δ_h, and subjective probabilities γ_s^h denoting the probability of reaching state s from its predecessor s^*, for all $s \in S \backslash \{0\}$. We assume that the utility function for consumption in each state $s \in S$, $u^h : R_+^L \rightarrow R$, is differentiable, concave, and weakly monotonic (more of every good is strictly better). The expected utility to agent h is

$$U^h = u^h(c_0) + \sum_{s \in S \backslash 0} \delta_h^{t(s)} \bar{\gamma}_s^h u^h(c_s). \tag{3}$$

where $\bar{\gamma}_s^h$ is the probability of reaching s fom 0 (obtained by taking the product of γ_σ^h over all nodes σ on the path $(0, s]$ from 0 to s).

Investor h's endowment of the consumption good is denoted by $e_s^h \in R_+^L$ in each state $s \in S$. His endowment of the assets at the beginning of time 0 is $y_{0^*}^h \in R_+^K$ (agents have initial endowment of assets only at the beginning). We assume that the consumption goods are all present, $\sum_{h \in H}(e_s^h + d_s y_{0^*}^h) >> 0, \forall s \in S$.

4.1.3 Production

We allow for durable consumption goods (inter-period production) and for intra-period production. For each $s \in S \backslash \{0\}$, let $F_s^h : \mathbb{R}_+^L \rightarrow \mathbb{R}_+^L$ be a concave inter-period production function connecting a vector of consumption goods at state s^* that h is consuming with the vector of consumption goods it becomes in state s. In contrast to consumption goods, it is assumed that all financial assets are perfectly durable from one period to the next, independent of who owns them.

For each $s \in S$, let $Z_s^h \subset \mathbb{R}^{L+K}$ denote the set of feasible intra-period production for agent h in state s. Notice, that assets and consumption goods can enter as inputs and outputs of the intra-period production process. Inputs appear as negative components of $z_i < 0$ of $z \in Z^h$, and outputs as positive components $z_i > 0$ of z.

4.1.4 Collateral and Contracts

Contract $j \in J$ is a contract that promises the consumption vector $j_{s'} \in R_+^L$ in each state s'. Each contract j defines its issue state $s(j)$, and the asset $k(j)$ used as collateral. We denote the set of contracts with issue state s backed by one unit of asset

k by $J_s^k \subset J$. We consider one-period contracts, that is, each contract $j \in J_s^k$ delivers only in the immediate successor states of s, i.e. $j_{s'} = 0$ unless $s' \in S(s)$. Contracts are defined extensively by their payment in each successor state. Notice that this definition of contract allows for promises with different baskets of consumption goods in different states. Finally, $J_s = \bigcup_k J_s^k$ and $J = \bigcup_{s \in S \setminus S_T} J_s$.

The price of contract j in state $s(j)$ is π_j. An investor can borrow π_j at $s(j)$ by selling contract j, that is by promising $j_{s'} \in R_+^L$ in each $s' \in S(s(j))$, provided he holds one unit of asset $k(j)$ as collateral.

Since the maximum a borrower can lose is his collateral if he does not honor his promise, the actual delivery of contract j in states $s' \in S(s(j))$ is $min\{q_{s'} \cdot j_{s'}, p_{s'k(j)} + q_{s'} \cdot d_{s'}^k\}$.

The *Loan-to-Value* LTV_j associated to contract j in state $s(j)$ is given by

$$LTV_j = \frac{\pi_j}{p_{s(j)k}}. \tag{4}$$

As before, the *margin* m_j associated to contract j in state $s(j)$ is $1 - LTV_j$. *Leverage* associated to contract j in state $s(j)$ is the inverse of the margin, $1/m_j$ and moves monotonically with LTV_j.

Finally, as in Section 3, we define the *average loan to value*, LTV for asset k, as the trade-value weighted average of LTV_j across all debt contracts actively traded in equilibrium that use asset k as collateral, and the *diluted average loan to value*, LTV_0^k (which includes assets with no leverage) by

$$LTV^k = \frac{\sum_h \sum_j \max(0, \varphi_j^h)\pi_j}{\sum_h \sum_j \max(0, \varphi_j^h) p_{s(j)k}} \geq \frac{\sum_h \sum_j \max(0, \varphi_j^h)\pi_j}{\sum_h y_{s(j)k}^h p_{s(j)k}} = LTV_0^k.$$

4.1.5 Budget Set

Given consumption prices, asset prices, and contract prices (q, p, π), each agent $h \in H$ choses intra-period production plans of goods and assets, $z = (z_c, z_y)$, consumption, c, asset holdings, y, and contract sales/purchases φ in order to maximize utility (3) subject to the budget set defined by

$B^h(q, p, \pi) = \{(z_c, z_y, c, y, \varphi) \in R^{SL} \times R^{SK} \times R_+^{SL} \times R_+^{SK} \times (R^{J_s})_{s \in S \setminus S_T} : \forall s$

$q_s \cdot (c_s - e_s^h - F_s^h(c_{s^*}) - z_{sc}) + p_s \cdot (y_s - y_{s^*} - z_{sy}) \leq$

$q_s \cdot \sum_{k \in K} d_s^k y_{s^*k} + \sum_{j \in J_s} \varphi_j \pi_j - \sum_{k \in K} \sum_{j \in J_{s^*}^k} \varphi_j min\{q_s \cdot j_s, p_{sk} + q_s \cdot d_s^k\}$;

$z_s \in Z_s^h$;

$\sum_{j \in J_s^k} max(0, \varphi_j) \leq y_s^k, \forall k\}$.

In each state s, expenditures on consumption minus endowments plus any produced consumption good (either from the previous period or produced in the current period), plus total expenditures on assets minus asset holdings carried over from previous periods and asset output from the intra-period technology, can be at most equal to total asset deliveries plus the money borrowed selling contracts, minus the payments due at s from contracts sold in the past. Intra-period production is feasible. Finally, those agents who borrow must hold the required collateral.

4.1.6 Collateral Equilibrium

A *Collateral Equilibrium* in this economy is a set of consumption good prices, financial asset prices and contract prices, production and consumption decisions, and financial decisions on assets and contract holdings $((q, p, \pi), (z^h, c^h, y^h, \varphi^h)_{h \in H}) \in (R_+^L)_{s \in S} \times (R_+^K \times R_+^{J_s})_{s \in S \setminus S_T} \times (R^{S(L+K)} \times R_+^{SL} \times R_+^{SK} \times (R^{J_s})_{s \in S \setminus S_T})^H$ such that

1. $\sum_{h \in H} (c_s^h - e_s^h - F_s^h(c_{s^*}) - z_{sc}^h) = \sum_{h \in H} \sum_{k \in K} y_{s^*k}^h d_s^k, \forall s$.
2. $\sum_{h \in H} (y_s^h - y_{s^*}^h - z_{sy}^h) = 0, \forall s$.
3. $\sum_{h \in H} \varphi_j^h = 0, \forall j \in J_s, \forall s$.
4. $(z^h, c^h, y^h, \varphi^h) \in B^h(q, p, \pi), \forall h$

   $(z, c, y, \varphi) \in B^h(q, p, \pi) \Rightarrow U^h(c) \leq U^h(c^h), \forall h$.

Markets for consumption, assets and promises clear in equilibrium and agents optimize their utility in their budget set.

4.2 General No-Default Theorem

It turns out that we can still assume no default in equilibrium without loss of generality in this much more general context as the following theorem shows.

Binomial No-Default Theorem:

Suppose that S is a binomial tree, that is S(s)={sU,sD} for each $s \in S\backslash S_T$. Suppose that all assets are financial assets and that every contract is a one period contract. Let $((q,p,\pi),(z^h,c^h,y^h,\varphi^h)_{h\in H})$ be an equilibrium. Suppose that for any state $s \in S\backslash S_T$, any asset $k \in K$, and any contract $j \in J_s^k$, the max min promise $(\bar{\lambda} j_{sU}, \bar{\lambda} j_{sD})$ is available to be traded, where $\bar{\lambda} = \max\{\lambda \in R_+ : \lambda(q_{sU}\cdot j_{sU}, q_{sD}\cdot j_{sD}) \leq (p_{sUk} + q_{sU}\cdot d_{sU},\ p_{sDk} + q_{sD}\cdot d_{sD})\}$. Then we can construct another equilibrium $((q,p,\pi),(z^h,c^h,\bar{y}^h,\bar{\varphi}^h)_{h\in H})$ with the same asset and contract prices and the same production and consumption choices, in which only max min contracts are traded.

Proof:

The proof of the Binomial No-Default Theorem can be applied in this more general context state by state, asset by asset, and ray by ray. Take any $s \in S\backslash S_T$ and any asset $k \in K$. Partition J_s^k into $J_s^k(r_1) \cup ... \cup J_s^k(r_n)$ where the r_i are distinct rays $(\mu_i,\nu_i) \in \mathbb{R}^2_+$ of norm 1 such that $j \in J_s^k(r_i)$ if and only if $(q_{sU}\cdot j_{sU}, q_{sD}\cdot j_{sD}) = \lambda(\mu_i,\nu_i)$ for some $\lambda > 0$. For each agent $h \in H$, consider the portfolio $(y^h(s,k,i),\varphi^h(s,k,i))$ defined by

$$\varphi_j^h(s,k,i) = \varphi_{sj}^h \text{ if } j \in J_s^k(r_i) \text{ and 0 otherwise.}$$
$$y^h(s,k,i) = \sum_{j\in J_s^k(r_i)} \max(0,\varphi_{sj}^h).$$

Denote the portfolio payoffs in each state by

$$w_U^h(s,k,i) = y^h(s,k,i)[p_{sUk} + q_{sU}d_{sU}^k] - \sum_{j\in J_s^k(r_i)} \varphi_j^h(s,k,i)\min(q_{sU}\cdot j_{sU}, p_{sUk} + q_{sU}d_{sU}^k).$$
$$w_D^h(s,k,i) = y^h(s,k,i)[p_{sDk} + q_{sD}d_{sD}^k] - \sum_{j\in J_s^k(r_i)} \varphi_j^h(s,k,i)\min(q_{sD}\cdot j_{sD}, p_{sDk} + q_{sD}d_{sD}^k).$$

If

$$\frac{\mu_i}{\nu_i} < \frac{p_{sUk} + q_{sU}d_{sU}^k}{p_{sDk} + q_{sD}d_{sD}^k}.$$

then the combination of the Arrow U security (which can be obtained by buying the asset k while borrowing on the max min contract of type (s,k,i)) and the max min contract of type (s,k,i) positively spans $(w_U^h(s,k,i), w_D^h(s,k,i))$. Thus we can apply the proof of the Binomial No-Default Theorem to replace all the above trades of contracts in $J_s^k(r_i)$ with a single trade of the max min contract of type (s,k,i). If

$$\frac{\mu_i}{\nu_i} > \frac{q_{sUk} + p_{sU} d_{sU}^k}{q_{sDk} + p_{sD} d_{sD}^k}$$

then exactly the same logic of the Binomial No-Default Theorem applies, but with the Arrow D security instead of the Arrow U security. If there is equality in the above comparison, then the contract and the asset are perfect substitutes, so there is no need to trade the contracts in the family at all.■

4.3 Discussion

The main idea of the proof is to apply the simple proof of Section 3 state by state to each asset and each homogeneous family of promises using the asset as collateral. Since every contract consists of a promise and its own collateral, the proof focuses on just the collateral needed to back the promises along a single ray. As in the proof in Section 3, the borrower can use less of this collateral to achieve the same final payoffs at the same cost by using only the max min contract. It may now be the case that sometimes the payoff cone is given by the positive span of the max min of the family and the Arrow D security, instead of the Arrow U security. However, the logic of the argument stays completely unaltered. Notice that despite the fact that we break the proof down ray by ray, general equilibrium interactions between different promises are a crucial part of the equilibrium.

The reader may realize that one can always rename any promise by its actual delivery and then one could trivially replace the original equilibrium by one in which there is no default. However, it is important to understand that our no-default theorem states something stronger: it shows that we can assume that the deliveries are on the same ray as the promises. One might have expected the possibility of default to open up a bigger span than that of the promises. What our theorem shows is that in binomial collateral models, the possibility of default does not lead to any improvement in the span of asset deliveries.

The No-Default Theorem allows for two further extensions. First, it can be extended to more than two successor states, provided that for each financial asset the states can be partitioned into two subsets on each of which the collateral value (including dividends of the asset) and the promise value of each contract written on the asset are constant. Second, it can also be extended to contracts with longer maturities.

Suppose all the contracts written on some financial asset come due in the same period and that the states in that period can be partitioned into two subsets on each of which the collateral value (including dividends of the asset) and the promise value of each contract written on the asset are constant. Suppose also that the financial asset used as collateral cannot be traded or used for production purposes before maturity. Then the proof of the Binomial No-Default Theorem shows that without loss of generality we can assume no default in equilibrium.

Let us now discuss how our other previous results extend to this more general framework. First, our two refinements studied in Section 3.3 extend to this general setting. Any positive fee for collateral use guarantees that in every equilibrium only max min contracts are traded. The extension relies on the fact that it is still the case that the max min element of each family is the contract that requires the least collateral.

Suppose now that we restrict attention to non-contingent contracts, that is contracts $j \in J_s^k$ for which in equilibrium $q_{sU} \cdot j_{sU} = q_{sD} \cdot j_{sD}$.[10] The Binomial Leverage theorem presented in Section 3.4 extends to many assets. Letting LTV_s^k denote the leverage of every riskless loan collateralized by k in state s, we must have

$$LTV_s^k = \frac{min\{p_{sD} + q_{sD} \cdot d_{sD}^k, p_{sU} + q_{sU} \cdot d_{sU}^k\}/p_{sk}}{1 + r_s}.$$

where $(1 + r_s) = q_{sU} \cdot j_{sU}/\pi_j$ for any non-contingent contract $j \in J_s$ whose deliveries are fully covered by the collateral. This formula explains which assets are easier to leverage. The asset whose future value has the least bad downside can be leveraged the most. The formula allows us to rank leverage of different assets at the same state s, or even across states and across economies.

When contract promises are contingent, the Binomial No-Default Theorem tells us exactly how big the promises of each type will be made per unit of collateral: as big as can be guaranteed not to default. But the leverage formula for the LTV associated to non-contingent contracts cannot so easily be extended to contingent contact promises. The non-contingent formula above does not require any information about the promises. With contingent promises one would need to know the ray on which the promise lies and also the state prices.

[10]These may be created by assuming there is some numeraire bundle of goods v_s such that commodity prices always satisfy $q_s \cdot v_s = 1$ and then supposing that promises are denoted in units of the numeraire. In the actual world, many contract promises are denoted by non-contingent money payments.

Finally, the State Pricing Corollary of section 3.1 does not extend to this more general context. For each ray, say r_i, we obtain (by the same logic as before), state prices a_i and b_i. However, they need not be the same as the state prices obtained when the argument is applied to a different ray, say r_j. The reason for this is that the payoff cones associated to each ray may not completely coincide. Hence, we only have a "local" state pricing result. The consequence of the failure of unique state pricing is that there is in general no single probability measure on the tree of states that allows us to rank asset leverage by looking at asset volatility. By contrast, the worst case return is a concept that is independent of probabilities.

5 Examples

In this section we present two examples in order to illustrate the theoretical results presented in Sections 3 and 4.

5.1 Binomial CAPM with Multiple Equilibria.

We assume one perishable consumption good and one asset which pays dividends $d_U > d_D$ of the consumption good. Consider two types of mean-variance investors, $h = A, B$, characterized by utilities $U^h = u^h(c_0) + \sum_{s \in S_T} \gamma_s u^h(c_s)$, where $u^h(c_s) = c_s - \frac{1}{2}\alpha^h c_s^2$, $s \in \{0, U, D\}$. Agents do not discount the future. Agents have an initial endowment of the asset, $y_{0^*}^h$, $h = A, B$. They also have endowment of the consumption good in each state, $e_s^h, \forall s, h = A, B$. It is assumed that all contract promises are of the form (j, j), $j \in J$, each backed by one unit of the asset as collateral. Agents will never deliver on a promise beyond the value of the collateral since we assume non-recourse loans.[11]

Suppose agents start with endowment of the asset, $y_{0^*}^A = 1, y_{0^*}^B = 3$. Suppose consumption good endowments are given by $e^A = (e_0^A, (e_U^A, e_D^A)) = (1, (1, 5))$ and $e^B = (e_0^B, (e_U^B, e_D^B)) = (3, (5, 5))$. Utility parameters are given by, $\gamma_U = \gamma_D = .5$ and $\alpha^A = .1$ and $\alpha^B = .1$. Finally, asset payoffs are $d_U = 1$ and $d_D = .2$. Type-A agents have a tremendous desire to buyArrow U securities and present consumption, and

[11]This example would satisfy all the assumptions of the classical CAPM (extended to untraded endowments), provided that we assumed agents always kept their promises, without the need of posting collateral.

Table 1: Collateral Equilibrium with No Default: Prices and Leverage.

Variable	Notation	Value
Asset Price	p	0.3778
State Price	a	0.3125
State Price	b	0.3264
Max min Contract Price	π_{j^*}	0.1278
Leverage	LTV^Y	0.3382

Table 2: Collateral Equilibrium with No Default: Allocations

	Asset and Collateral		
	Asset y	Contracts φ_{j^*}	
Type-A	3.7763	3.7763	
Type-B	0.2237	-3.7763	
	Consumption		
	$s=0$	$s=U$	$s=D$
Type-A	0.4337	4.0211	5
Type-B	3.5663	5.9789	5.80

to sell Arrow D securities. But they are limited by the restriction to non-contingent contract promises (j, j).

According to the Binomial No-Default Theorem, in searching for equilibrium we never need to look beyond the max min promise $j^* = .2$, for which there is no default. Tables 1 and 2 present this max min collateral equilibrium. Type-A agents buy most of the assets in the economy, $y^A = 3.7763$, and use their holdings as collateral to sell the max min contract, promising (.2)(3.7763) in both states U and D. Type-B investors sell most of their asset endowment and lend to type-A investors.[12]

As indicated by the Binomial State Pricing Corollary, all the contracts $j \neq j^*$, as

[12] To find the equilibrium we guess the regime first and we solve for three variables, p, π_{j*} and ϕ_{j*}, a system of three equations. The first equation is the first order condition for lending corresponding to the risk averse investor:$\pi = \frac{q_U(1-\alpha^A c_U^A)d_D + q_D(1-\alpha^A c_D^A)d_D}{1-\alpha^A c_0^A}$.The second equation is the first order condition of the tolerant investor for purchasing the asset via the max min contract, $p - \pi = \frac{q_U(1-\alpha^T c_U^T)(d_U - d_D) + q_D(1-\alpha^T c_D^T)(d_D - d_D)}{1-\alpha^T c_0^T}$. The third equation is the first order condition for the risk averse investor for holding the asset, $p = \frac{\gamma_U(1-\alpha^A c_U^A)d_U + \gamma_D(1-\alpha^A c_D^A)d_U}{1-\alpha^A c_0^A}$. Finally, we check that the regime is genuine, confirming that the tolerant investor really wants to leverage to the max, for this to be the case, $\pi > \frac{\gamma_U(1-\alpha^T c_U^T)d_D + \gamma_D(1-\alpha^T c_D^T)d_D}{1-\alpha^T c_0^T}$.

Table 3: Collateral Equilibrium with Default: Prices and Leverage

Variable	Notation	Value
Asset Price	p	0.3778
Promise	j	0.2447
Contract j Price	π_j	0.1418
Leverage	LTV^Y	0.3753

well as $j = j^*$, can be priced by state prices $a = 0.3125$ and $b = 0.3264$. By the No-Default Theorem, we do not need to investigate trading in any of the contracts $j \neq j^*$. Indeed it is easy to check that this is a genuine equilibrium, and that no agent would wish to trade any of these contracts $j \neq j^*$ at the prices given by a, b. Every agent who leverages chooses to sell the same max min contract, hence asset leverage and contract leverage are the same and described in the table. We can easily check that the LTV^Y satisfies both formulas given in the Binomial Leverage Theorem and the Binomial Leverage-Volatility Theorem. So that

$$LTV^Y = \frac{d_D/p}{1 + r_{j^*}} = \frac{.2/.3778}{1.56} = .3382.$$

$$1 - LTV^Y = m = \sqrt{\frac{\alpha}{\beta} \frac{Vol_{\alpha,\beta}(Y)}{(1 + r_{j^*})p}} = \sqrt{\frac{.80}{.83} \frac{0.6553}{(1.56).3778}} = .6618$$

This equilibrium is essentially unique, but not strictly unique. In fact, there is another equilibrium *with* default as shown in Tables 3 and 4, in which the type-A agents borrow by selling the contract $j = .2447 > j^* = .2$. In the default equilibrium, leverage is higher and the asset holdings of type-A agents are higher (so diluted leverage is much higher). They borrow more money. However, as guaranteed by the Binomial Default Theorem, in both equilibria, consumption and asset and contract prices are the same: *actual* default is irrelevant. Notice that in the no-default equilibrium, 3.7763 units of the asset are used as collateral, while in the default equilibrium 4 units of the asset are used as collateral.The no-default equilibrium uses less collateral.

Between these two equilibria, the Modigliani-Miller Theorem holds; there is an indeterminacy of debt issuance in equilibrium. However, leverage cannot be reduced below the max min contract level. If type-A agents were forced to issue still less debt, they would rise in anger. Thus in this example, the No-Default Theorem holds

Table 4: Collateral Equilibrium with Default: Allocations.

	Asset and Collateral		
	Asset y	Collateral φ_j	
Type-A	4	4	
Type-B	0	-4	
	Consumption		
	$s = 0$	$s = U$	$s = D$
Type-A	0.4337	4.0211	5
Type-B	3.5663	5.9789	5.80

Table 5: Arrow-Debreu and CAPM Equilibrium.

Asset Price	p	0.3700	
State Price	p_U	0.3125	
State Price	p_D	0.2875	
	Consumption		
	$s = 0$	$s = U$	$s = D$
Type-A	0.5338	4.0836	4.5569
Type-B	3.4662	5.9164	6.2431
	CAPM Portfolios:	Market	Bond
Type-A		0.5916	1.8328
Type-B		0.4084	-1.8328

while the Modigliani-Miller Theorem fails beyond a limited range.

Finally, both collateral equilibria are different from the Arrow-Debreu Equilibrium and the classical CAPM equilibrium as shown in Table 5.

State prices in collateral equilibrium are different from the state prices in Arrow-Debreu equilibrium. The asset price in complete markets is slightly lower than in collateral equilibrium. In the complete markets equilibrium, investors hold shares in the market portfolio $(10, 10.8)$ (aggregate endowment) and in the riskless asset $(1, 1)$.

5.2 Binomial CAPM with Unique Equilibrium.

Consider the same CAPM model as before but with the following parameter values. Suppose agents each own one unit of the asset, $y^h_{0^*} = 1, h = A, B$. Suppose consumption good endowments are given by $e^A = (e^A_0, (e^A_U, e^A_D)) = (1, (1, 5))$ and

Table 6: Collateral Equilibrium with No Default: Prices and Leverage.

Variable	Notation	Value
Asset Price	p	0.4572
State Price	a	0.4027
State Price	b	0.2725
Max min Contract Price	π_{j^*}	0.1350
Leverage	LTV^Y	0.2952

Table 7: Collateral Equilibrium with No Default: Allocations

	Asset and Collateral		
	Asset y	Contracts φ_{j^*}	
Type-A	2	2	
Type-B	0	-2	
	Consumption		
	$s=0$	$s=U$	$s=D$
Type-A	0.8122	2.6	5
Type-B	3.1872	5.4	5.4

$e^B = (e_0^B, (e_U^B, e_D^B)) = (3, (5, 5))$. Utility parameters are given by, $\gamma_U = \gamma_D = .5$ and $\alpha^A = .1$ and $\alpha^B = .1$. Finally, asset payoffs are $d_U = 1$ and $d_D = .2$.

Tables 6 and 7 present the max min collateral equilibrium. In the collateral equilibrium type-A agents buy all the asset in the economy and use all of their holdings as collateral, leveraging via the max min contract. On the other hand, type-B investors sell all their asset endowment and lend. As before LTV^Y is characterized by tail risk and volatility formulas.

Unlike the previous example the no-default equilibrium in this example is unique without any need of refinement. We cannot find another equilibrium involving default with borrowers issuing bigger promises, since there is not enough collateral in the economy. In this case, as before, the collateral equilibrium does not coincide with the complete markets equilibrium shown in Table 8.

Table 8: Arrow-Debreu and CAPM equilibrium

Asset Price	p	0.4350	
State Price	p_U	0.3750	
State Price	p_D	0.3	
	Consumption		
	$s = 0$	$s = U$	$s = D$
Type-A	0.8024	3.1018	4.4814
Type-B	3.1976	4.8982	5.9186
	CAPM Porfolios	Market	Bond
Type-A		0.5749	1.4970
Type-B		0.4251	−1.4970

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