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Sumit Joshi
George Washington University

Ahmed Saber Mahmud
Johns Hopkins University

Sudipta Sarangi
Virginia Tech

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Institute for International Economic Policy
1957 E St. NW, Suite 502
Voice: (202) 994-5320
Fax: (202) 994-5477
Email: iiep@gwu.edu
Web: www.gwu.edu/~iiep

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Sumit Joshi[†]
George Washington University

Ahmed Saber Mahmud[‡]
Johns Hopkins University

Sudipta Sarangi[§]
Virginia Tech

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Abstract

Economic agents are typically connected to others in multiple network relationships, and the architecture of one network could be shaped by connections in other networks. This paper examines the formation of one network when connections in a second network are inherited under two scenarios: (i) the inherited network is asymmetric allowing for a wide range of graphs called nested split graphs, and (ii) the inherited network is a symmetric type of network belonging to a subclass of regular graphs. Both the inherited and endogenously formed networks are interdependent because the respective actions in each are (weak) strategic complements. This property is sufficient to show that those who inherit high centrality will continue to have high centrality. Additionally, the network formed by the agents induces a coarser partition than the inherited network, suggesting the possibility of being able to improve network centrality, but only in a limited manner. Thus, our analysis explains preferential attachment and why inequality is often entrenched in society, how asymmetries in one network may be magnified or diminished in another, and what determines the identity of players occupying the various vertices of asymmetric equilibrium networks.

JEL classification: C72, D85

Key words and phrases: Network formation, multigraphs, strategic complementarities, Katz-Bonacich centrality, nested split graphs.

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[†]Department of Economics, George Washington University, 371 Monroe Hall, 2115 G Street NW, Washington DC 20052, USA. Phone: 202-994-6154; Email: sumjos@gwu.edu.

[‡]Applied Economics Program, Johns Hopkins University, Suite 104, 1717 Massachusetts Avenue, Washington DC 20036, Email: amahmud2@jhu.edu.

[§]Department of Economics, 3016 Pamplin Hall, Virginia Tech, Blacksburg, VA 24061, Email: ssarangi@vt.edu.

1 Introduction

The typical economic agent is concurrently engaged in many different relationships. Individuals can simultaneously belong to both social and business networks, firms can be involved in multiple collaborative relationships along different dimensions of R&D, and countries can be signatories to multiple economic and/or military alliances. This raises an important question: When agents straddle multiple networks, then to what extent are an agent's connection in one network shaped by the connections in others? This question has largely been left unaddressed in the literature[1] which has attempted to explain the architecture of equilibrium networks in terms of the benefits and costs of links established *within* a network, without any reference to the role of other networks. In this paper, we allow for the co-existence of multiple network relationships and delineate the role of one network in shaping the equilibrium architecture of another.

There is a substantial body of empirical evidence to support the dependence of one network of relationships on other networks.[2] For instance, social networks can have a strong influence on business networks. Belonging to specific clubs, being a member of certain social groups or attending certain events in the real or virtual world can play a key role in business affairs. This link between social and business relationships has been well documented for the *guanaxi* in China (Kali, 1999), the *chaebols* in Korea, and several communities like the Marwaris and Parsis in India (Damodaran, 2008). This has even been studied in the context of networks by examining marriage alliances. Padget and Ansell (1993) examine marriages between 16 elite families in Florence during the early 1400s to explain the emergence of the Medici as an economic force. They document that the Medici's rise in economic and political circles can be traced to their increasing importance in the marriage network and the advantages it conferred with respect to communicating information, brokering business deals, and reaching political decisions. Similarly, Munshi (2011) demonstrates that a tight community network can help a poor community leap-frog ahead as long as some members of the disadvantaged community have access to other networks of opportunities.

There is a significant body of work outside economics that deals with multiple networks, also referred to as *multiplex* networks.[3] Much of this literature is rooted in computer science, is simulation driven, and does not explicitly model strategic issues.[4] The paper in the economics literature that is closest to our study

[1] In recent years networks have emerged as an important tool for modeling the interaction structure among a set of agents. Following the lead of Jackson and Wolinsky (1996) and Bala and Goyal (2000), there is now a large literature on the endogenous formation of networks in contexts ranging from labor markets, homophily, and peer pressure to trade, crime, team work, and R&D cooperation. Please see Goyal (2007) and Jackson (2008) for more details.

[2] Powell and Padgett (2012) even provide a taxonomy of multiple types of relations in networks as follows. They claim that on one hand similar individuals are engaged in the same relationship, but with people from different social worlds, like one's Facebooks friends consisting of individuals from elementary school as well as from college or work. On the other hand, they say different individuals with their own networks may be brought into contact at through a social or economic institution, like an interdisciplinary hiring initiative at a university.

[3] In contrast to the term multiplex networks which primarily focus on networks of different relationships, we have chosen to use the term multigraph to capture the notion of network formation.

[4] Often the focus is on issues like identifying communities in exogenously given multiplex networks (Mondragon et al. (2017)) or examining co-evolution of multiplex networks based on a pre-specified feedback mechanism (Wu et al. 2017). Although there are a few papers examining the evolution of cooperation in multiplex networks (Gómez-Gardeñes et al. (2012)), they usually assume that the networks are exogenously given. Similarly, there are papers relating to multiplex interbank networks, which are again devoid of strategic elements (Bargigle et al. (2015)).

is recent work of Chen et al. (2016). Players in their model engage in two different activities which can be complements or substitutes and therefore can be seen as two different types of relationships. However, the authors take the network as given, and do not address the network formation problem. Apart from the focus on network formation, we differ from this literature in two main ways: (i) by assuming that the networks are not exogenously given, and (ii) by incorporating strategic behavior.

Our paper develops a formal "multigraph" model to explain these types of situations. We assume that agents are participants in two networks. To fix ideas let us call one of these the network of *social connections* (denoted by $\mathbf{H}$), and the other the network of economic activities or the *business network* (denoted by $\mathbf{G}$). Each agent's utility is a function of what happens in both networks. Following Ballester, Calvo-Armengol, and Zenou (2006), the utility function of each agent exhibits (local) complementarities in effort with whom the player is linked within a network. Moreover, in order to capture the interaction between networks, we assume that the effort levels of a player across networks are also complementary. This captures the notion that the amount of time and energy devoted towards social and business networks are strategic complements. For instance, a person who is well known in her social network may be able to secure credit on better terms, or have access to better business opportunities as in many of the cases mentioned above. Biggs, Raturi and Shrivastav (2002) find that in Kenya, a borrower's membership in ethnic networks does not play a role in access to commercial credit, but is important for supplier credit, especially among Asian groups. They argue that this is due to the fact that ethnic groups have better information and contract enforcement mechanisms. Fafchamps (2000) documents a similar outcome using data from manufacturing firms in Kenya and Zimbabwe. Clearly, the efforts exerted in these cases in the two networks are complementary.

In our analysis, we assume that the network of social connections, $\mathbf{H}$, is fixed. An intuitive way to think about this is that agents inherit their social connections. We believe that this is a realistic assumption that also makes multigraph network formation tractable from an analytical perspective. Alternatively, this can be also viewed as a way to understand the role of feedback effects when the formation of multiple networks is not simultaneous but asynchronous. In other words, it allows us to capture the role of path dependence in network formation making it especially useful to understand questions about inequality in networks. Hence in our model, link formation occurs only in the network $\mathbf{G}$, but the results show how the social network $\mathbf{H}$ affects link formation in $\mathbf{G}$. We view the network formation game in $\mathbf{G}$ as a two-stage process. In the first stage players form their network $\mathbf{G}$, and in the second stage the agents exert effort in both networks resulting in payoffs. The second stage game is a standard Nash game in efforts, conditional on the networks from the first stage. The first stage network formation game is modeled as a sequential game starting from the empty network where the order in which agents make their link decisions is predetermined. When it is a agent's turn to move, she can delete (any number of) links as well as propose one (and only one) link in $\mathbf{G}$ to an unlinked agent; the link is formed if the potential partner accepts the proposal. Agents in the game move repeatedly till no agents wishes to make any more moves. The network formation game is shown to generate an "improving path" of networks which converges to a pairwise-stable equilibrium. The *pairwise-stable equilibrium*, proposed by Goyal and Joshi (2006), corresponds to a network $\mathbf{G}$ in which no agent has an incentive to delete links and no pair of unlinked agents can mutually profit from a link.

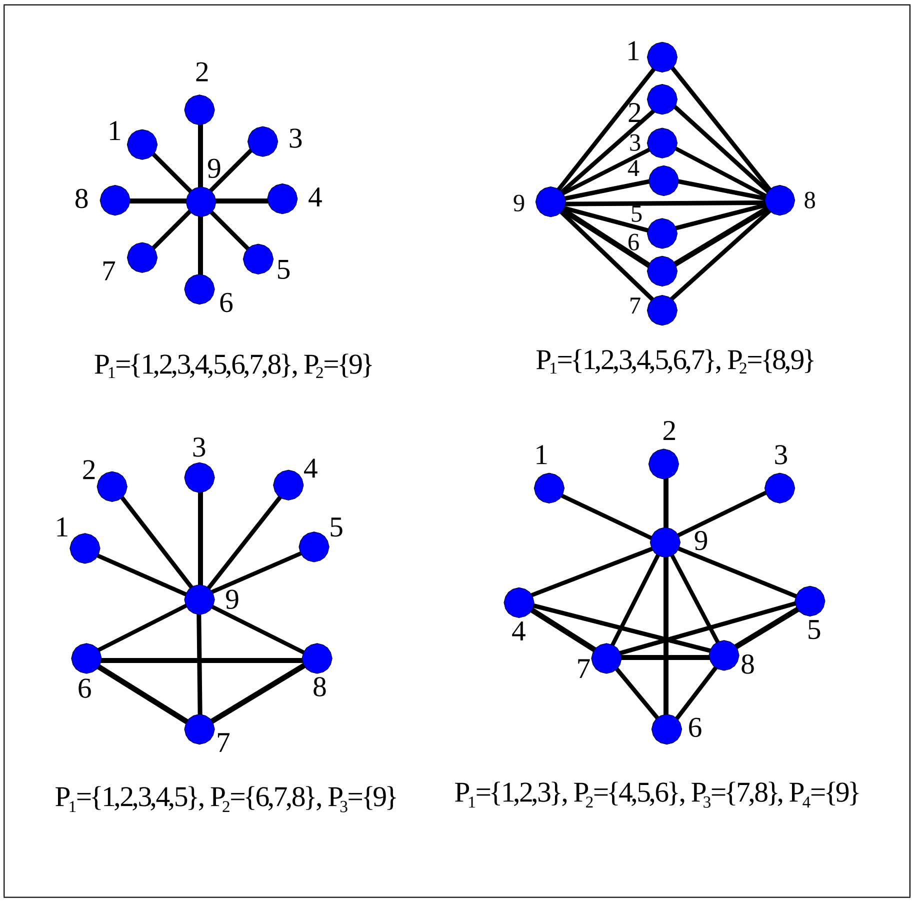

Figure 1: Nested Split Graphs with their Degree Partitions

Strategic complementarities in actions, both within and across networks, create feedback loops that need to be bounded to ensure the existence of a Nash equilibrium in the second stage. This requires that the degree of complementarity in actions within networks as well as across the two networks be sufficiently small. Therefore the linkage between the two networks, operating through complementarity of actions, is relatively weak. Our result however is quite striking. In spite of the relatively tenuous connection between the two networks, the architecture of $\mathbf{H}$ has an important bearing on the equilibrium structure of $\mathbf{G}$. To draw out this implication we consider two extreme architectures for the fixed network $\mathbf{H}$: *Nested split graphs* (or NSG), and a subclass of *regular* graphs. In a NSG the neighborhoods, or the set of direct connections of agents, are nested allowing for a large class asymmetric networks. Suppose we partition a network into groups using the degrees or number of direct connections of the players, then one way to think about a NSG is that the neighborhood of every agent in a particular partition subsumes the neighborhood of all the agents in partitions below (i.e. sets of agents with fewer degrees). According to this definition, either agents i and j are not connected, or they are friends, or friends of friends and so on. The star network for instance is a NSG. Examples are provided in figure 1 along with their respective degree partitions. The other kind of network we consider is a subclass of *regular graphs* in which all players have the same Katz-Bonacich centrality (henceforth *KB-regular* for brevity) giving rise to a rather symmetric architecture. Recall that regular graphs are graphs where every node has the same degree (or number of direct links). Intuitively, our subclass of KB-regular graphs requires that every node has the same level of importance in the network. The wheel, or cycle, network for example is KB-regular.

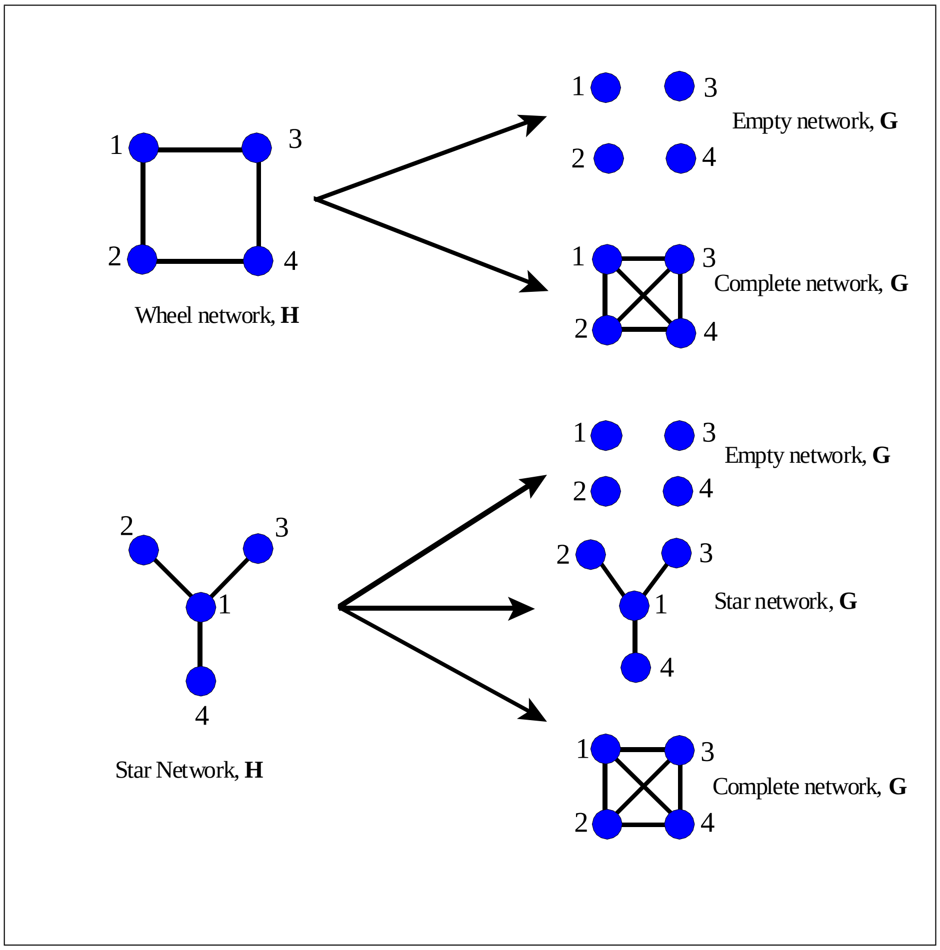

Figure 2: Impact of Network **H** on Network **G**

Our main result is the following. If the inherited network **H** is a NSG, then the endogenously formed network **G** is either empty, complete, or a NSG with at most one (non-singleton) component (Proposition 4). Moreover, the agents who have the most social connections in **H** also have the most connections in **G**, and those with the fewest connections in **H** have the fewest connections in **G**. We call this type of preferential attachment the *silver spoon effect*, which says that those who are privileged in terms of their social connections benefit more in the business network as well. On the other hand, when **H** is a KB-regular graph, then **G** is also regular and is either empty or complete (Proposition 5). These are illustrated in figure 2. By taking the interrelationship between **G** and **H** into account through (weak) strategic complementarities in actions, we provide an explanation for how inequality among agents maybe entrenched, with inequality in one network of connections (or an asymmetric network like NSG) being mirrored across other network connections as well. In a development context, our analysis sheds light on how class structures can be preserved in societies. Those born into privileged classes with high social connections have access to better business opportunities allowing them to do better. This may also explain why those with fewer connections may find it harder to have access to a larger set of economic opportunities and find it difficult to escape poverty traps.

Our main result can be explained as follows. The reduced form incremental utility of an agent from forging a link is a function of the partner's Katz-Bonacich centrality[5] in network **H** and the neighborhood in network **G** (Lemma 1). In the sequential network formation game, this creates an incentive for each agent to link to a partner with the highest Katz-Bonacich centrality in **H**. Starting from the empty network, this preferential attachment generates an improving path along which the most central agents in **H** have neighborhoods in **G** that nest the neighborhoods of less central players in **H**. Strategic complementarity obviates the incentive to delete any existing links and therefore the improving path converges to a limiting pairwise-stable equilibrium network that is also a NSG.

Our analysis addresses an open question in the network formation literature, namely predicting the *identity* of agents occupying various nodes in an asymmetric equilibrium network. The literature to date takes agents as ex-ante identical for reasons of tractability. But this implies that the equilibrium architecture is invariant to any permutation of agents' labels. By taking the influence of **H** into account we are able to characterize both the equilibrium network structure of **G** *and* the identity of agents occupying the vertices of the network. Our analysis shows that an agent's position in network **G** is grounded in her position in the architecture of network **H**. We can therefore make definite predictions regarding how the centrality of agents in the **H** network translates into their location in the **G** network.

Our analysis also offers perspective on whether inequalities are magnified or diminished in the process of network formation. If **H** is a NSG, then **G** assumes a *coarser* NSG architecture. Whether this coarser architecture corresponds to a higher or lower level of inequality in the number of direct connections (degree) depends on the cost of link formation. We show that when linking costs exceed a certain threshold, then then the degree distribution of **G** "shifts down" in the sense of first order dominance relative to **H** (Proposition 6). To the extent that players at the lower extreme of the degree distribution see a strict decrease in the number of their direct connections, the inequality inherent in **H** is amplified. The opposite is true when linking costs fall below a certain threshold.

[5] The Katz-Bonacich centrality of an agent counts the number of weighted walks in the network **H** originating from the agent, with the weights on the walks falling exponentially with their length. It is a measure of the prominence of the agent in network **H**.

The paper is organized as follows. Section 2 is a review of networks while section 3 presents the strategic complementarities model. The characterization of equilibrium networks is provided in section 4. Section 5 discusses the issue of unequal distribution of links. Section 6 provides applications of the multigraph model, and section 7 concludes. All proofs are collected in an appendix.

2 Preliminaries

Let $\mathcal{N} = \{1, 2, ..., N\}$ denote the set of players. We begin by reviewing the network concepts that will be used in the paper.

Graphs
A *network (*equivalently *graph*) is a tuple $(\mathcal{N}, \mathbf{H})$ where $\mathcal{N}$ is also referred to as nodes, and $\mathbf{H}$ records the undirected links that exist between the players. When the set of players is unambiguous, we refer to $\mathbf{H}$ as the network, and represent it in two alternative ways. The first is by letting $\mathbf{H}$ denote the collection of all pairwise links with $ij \in \mathbf{H}$ indicating that players i and j are linked in the network. The second is by letting $\mathbf{H} = [h_{ij}]$ denote the (symmetric) adjacency matrix such that $h_{ij} = h_{ji} = 1$ if i and j are linked and $h_{ij} = h_{ji} = 0$ otherwise. We assume that $h_{ii} = 0\ \forall i \in \mathcal{N}$, i.e. self-loops in a network are ruled out. We use both representations interchangeably.

The set $\mathbf{N}_i(\mathbf{H}) = \{j \in \mathcal{N}\backslash\{i\} : ij \in \mathbf{H}\}$ denotes the *neighbors* of player i in $\mathbf{H}$ and $d_i(\mathbf{H}) = |\mathbf{N}_i(\mathbf{H})|$ the cardinality of this set, also referred to as player i's *degree*. The vector $\mathcal{D}(\mathbf{H}) = \{d_1(\mathbf{H}), d_2(\mathbf{H}), ..., d_N(\mathbf{H})\}$ provides a listing of the degree of each player in the network $\mathbf{H}$. A *walk* in $\mathbf{H}$ connecting i and j is a set of nodes $\{i_1, \ldots, i_n\}$ such that $ii_1, i_1i_2, ..., i_{n-1}i_n, i_nj \in \mathbf{H}$. A *chain* is a walk in which all nodes are distinct.[6] A graph is *connected* if there exists a chain between any pair $i, j \in \mathcal{N}$; otherwise the network is *unconnected*. A sub-network, $C(\mathbf{H}) \equiv (\mathcal{N}', \mathbf{H}')$, $\mathcal{N}' \subset \mathcal{N}$, $\mathbf{H}' \subset \mathbf{H}$, is a *component* of the network $(\mathcal{N}, \mathbf{H})$ if it is connected, and if $ij \in \mathbf{H}$ for $i \in \mathcal{N}'$, $j \in \mathcal{N}$, implies $j \in \mathcal{N}'$ and $ij \in \mathbf{H}'$. Let $\mathbf{H} - ij$ (respectively $\mathbf{H} + ij$) denote the network obtained from $\mathbf{H}$ by deleting (respectively adding) the link ij.

In a *regular graph,* $d_i(\mathbf{H})$ is the same for all players, which is also the degree of the network. Examples include the complete network, $\mathbf{H}^c$, which has degree $N - 1$, and the empty network, $\mathbf{H}^e$, which has degree 0. A network shaped like a wheel, or cycle, is regular with degree 2.

Next, we define a *nested split graph* (NSG) along the lines of König et al. (2014). Consider a network $\mathbf{H}$ and let its distinct positive degrees be $d_{(1)} < d_{(2)} < \cdots < d_{(m)}$; let $d_{(0)} = 0$ even though there may not exist an isolated player in $\mathbf{H}$. Let $P_k(\mathbf{H}) = \{i \in \mathcal{N} : |\mathbf{N}_i(\mathbf{H})| = d_{(k)}\}$, $k = 0, 1, ..., m$. Denote the *degree partition* of $\mathbf{H}$ as $\mathcal{P}(\mathbf{H}) = \{P_0(\mathbf{H}), P_1(\mathbf{H}), ..., P_m(\mathbf{H})\}$. Let $\lfloor x \rfloor$ denote the largest integer smaller than or

[6] A walk with distinct nodes is also called a *path*. We follow Jackson and Watts (2002) in using the term path when referring to a sequence of networks.

equal to x. A network $\mathbf{H}$ is a NSG if for each player $i \in P_k(\mathbf{H})$, $k = 1, 2, .., m$:

$$\mathbf{N}_i(\mathbf{H}) = \begin{cases} \cup_{l=1}^{k} P_{m+1-l}(\mathbf{H}), & k = 1, 2, ..., \lfloor m/2 \rfloor \\ \cup_{l=1}^{k} P_{m+1-l}(\mathbf{H}) \backslash \{i\}, & k = \lfloor m/2 \rfloor + 1, ..., m \end{cases}$$

A *star* is a special case of a NSG where $|P_1(\mathbf{H})| = N - 1$ and $|P_m(\mathbf{H})| = 1$.

Katz-Bonacich Centrality

Letting $\mathbf{I}$ denote the identity matrix, and $\xi > 0$ an (sufficiently small) attenuation parameter, consider the matrix:

$$\mathbf{M}(\mathbf{H},\xi) \equiv [\mathbf{I} - \xi \mathbf{H}]^{-1} = \sum_{s=0}^{\infty} \xi^s \mathbf{H}^s \tag{1}$$

where $\mathbf{H}^0 = \mathbf{I}$. It is well known that since $\mathbf{H}$ is symmetric, all its eigenvalues are real and sum to zero, and thus the largest eigenvalue $\mu_{\max}(\mathbf{H})$ is positive. Further $[\mathbf{I} - \xi \mathbf{H}]^{-1}$ is well-defined and non-negative if $\xi \mu_{\max}(\mathbf{H}) < 1$. Given a $N \times 1$ vector $\mathbf{a}$, the vector of $\mathbf{a}$-weighted *Katz-Bonacich* (KB) *centralities* (Bonacich, 1987) of the players is given by:

$$\mathbf{b}(\mathbf{H},\xi, \mathbf{a}) = \mathbf{M}(\mathbf{H},\xi)\mathbf{a} = \sum_{s=0}^{\infty} \xi^s \mathbf{H}^s \mathbf{a} \tag{2}$$

When $\mathbf{a} = \mathbf{1}$, we get the unweighted KB centralities $\mathbf{b}(\mathbf{H},\xi, \mathbf{1})$ for the players. The i^{th}-component of the vector, $b_i(\mathbf{H},\xi, \mathbf{1})$, measures the number of ξ-weighted walks in the network $\mathbf{H}$ originating from node i with the weights on the walks falling exponentially with their length.[7]

KB-regular Graphs

To contrast the impact on network formation when the inherited network $\mathbf{H}$ is a highly asymmetric NSG, we will consider as a counterpoint the case where $\mathbf{H}$ is highly symmetric. In particular, we will consider a subclass of regular networks called *KB-regular* networks, in which all players have the same KB centrality.

[7] There is an interesting relationship between NSGs and KB-centrality. König et al. (2014, Proposition 1) show that for a NSG, the partition of players according to their KB centralities also coincides with their degree partition.

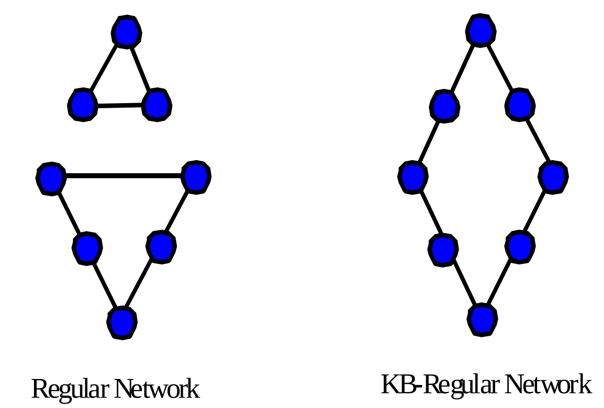

Figure 3: Regular and KB-Regular Networks

Figure 3 shows two networks for the case $N = 8$. The network on the left is regular since all players have the same degree (equal to 2). However, it is not KB-regular since all players do not have the same KB centrality. In contrast, the network on the right is KB-regular.

Pairwise Stable Equilibrium In our model players interact with each other through two different sets of connections. We will use the network $\mathbf{G}$ to keep track of players' links along one sphere, say economic, while network $\mathbf{H}$ accounts for links along another dimension, say the social. Thus, in our model, interactions between the players are captured by two separate adjacency matrices.

We consider a two-stage game. In Stage 1, we assume that: (i) network $\mathbf{H}$ is historically given and does not change, and (ii) players form the network $\mathbf{G}$. In the second stage, players play a Nash game contingent on $\mathbf{G}$ and $\mathbf{H}$. Let some closed set $A_i \subset \mathbb{R}^2_+$ denote the action set of player i for the second stage Nash game where an action is a vector of the form $a_i = (x_i, y_i) \in A_i$. Let $A = \times_{j=1}^{n} A_j$ denote the set of action profiles and $u_i : A \times \mathbf{G} \times \mathbf{H} \to \mathbb{R}$ the utility function of player i. Given networks $\mathbf{G}$ and $\mathbf{H}$, an action profile $\mathbf{a}^*(\mathbf{G})$ is a Nash equilibrium of the second stage game if :

$$u_i(a_i^*(\mathbf{G},\mathbf{H}), \mathbf{a}^*_{-i}(\mathbf{G},\mathbf{H}), \mathbf{G},\mathbf{H}) \geq u_i(a_i(\mathbf{G},\mathbf{H}), \mathbf{a}^*_{-i}(\mathbf{G},\mathbf{H}), \mathbf{G},\mathbf{H}), \quad \forall a_i \in A_i, \forall i \in \mathcal{N} \tag{3}$$

where $\mathbf{a}^*_{-i}(\mathbf{G},\mathbf{H})$ is the Nash action profile of players other than i. The (first stage) reduced form utility function of player i is given by:

$$U_i(\mathbf{G},\mathbf{H}) = u_i(a_i^*(\mathbf{G},\mathbf{H}), \mathbf{a}^*_{-i}(\mathbf{G},\mathbf{H}), \mathbf{G},\mathbf{H}) \tag{4}$$

Let $c \geq 0$ denote the cost to a player of a bilateral link in $\mathbf{G}$. Recall that $\mathbf{H}$ is historically given, therefore any linking costs associated with the formation of $\mathbf{H}$ are assumed to be sunk. The *net* utility function of player i from establishing the network $\mathbf{G}$ is given by $U_i(\mathbf{G}, \mathbf{H}) - d_i(\mathbf{G})c$. Note that this is also player i's net payoff from the game. Following Goyal and Joshi (2006) we will say that $\mathbf{G}$ is a *pairwise stable equilibrium* (or *pws-equilibrium*) if: (i) no player has an incentive to delete any subset of own links, and (ii) no pair of unlinked players i and j can profit by forming a link, i.e.

$$U_i(\mathbf{G} + ij, \mathbf{H}) - U_i(\mathbf{G}, \mathbf{H}) > c \ \Rightarrow \ U_j(\mathbf{G} + ij, \mathbf{H}) - U_j(\mathbf{G}, \mathbf{H}) < c \tag{5}$$

Network Formation and Improving Paths

To simplify the exposition, assume without loss of generality that players are indexed according to their degree in network $\mathbf{H}$. Recall that $\mathcal{P}(\mathbf{H}) = \{P_0(\mathbf{H}), P_1(\mathbf{H}), ..., P_m(\mathbf{H})\}$ denotes the degree partition of $\mathbf{H}$. Therefore, if $i \in P_l(\mathbf{H})$, $j \in P_{l'}(\mathbf{H})$, and $i < j$, then $l \leq l'$. The process starts in round 0 with the empty network, $\mathbf{G}(0) \equiv \mathbf{G}^e$. There are potentially infinitely many rounds[8], and in each round players move sequentially in the order of their index to propose changes to the network. In a given round, the player with the move is called the *active* player. In round $t = 1, 2, ...$, the active player i observes the current network $\mathbf{G}(t-1)$, and makes two decisions: (i) delete any subset of own links, and (ii) possibly propose a link to some unlinked player j.[9] Player j, to whom the link is proposed, is the *passive* player whose reactive role in round t is simply to reciprocate or reject the proposed link ij. The link ij is formed only if player j acquiesces. Any unilateral deletions of own links by the active player, and the possible formation of a new link with a passive player, leads to a network $\mathbf{G}(t)$ at the end of stage t. We will say in this case that $\mathbf{G}(t)$ is *reachable* from $\mathbf{G}(t-1)$. For example, if player i does not wish to delete any links, and j repudiates the proposed link ij, then $\mathbf{G}(t) = \mathbf{G}(t-1)$. Or, if player i does not wish to delete any links, and j reciprocates the proposed link ij, then $\mathbf{G}(t) = \mathbf{G}(t-1) + ij$. Thus both $\mathbf{G}(t-1)$ and $\mathbf{G}(t-1) + ij$ are reachable from $\mathbf{G}(t-1)$.

Following Jackson and Watts (2002), a sequence of networks, $\{\mathbf{G}(0), \mathbf{G}(1), .., \mathbf{G}(t), ..\}$ is an *improving path*[10] if the following three conditions hold: (i) $\mathbf{G}(t)$ is reachable from $\mathbf{G}(t-1)$, $t = 1, 2, ...$; (ii) the active player i in round t is strictly better off by the transition from $\mathbf{G}(t-1)$ to $\mathbf{G}(t)$:

$$U_i(\mathbf{G}(t), \mathbf{H}) - d_i(\mathbf{G}(t))c > U_i(\mathbf{G}(t-1), \mathbf{H}) - d_i(\mathbf{G}(t-1))c \tag{6}$$

and, (iii) if the passive player j reciprocates the proposed link so that $ij \in \mathbf{G}(t)$, then:

$$U_j(\mathbf{G}(t), \mathbf{H}) - d_j(\mathbf{G}(t))c \geq U_j(\mathbf{G}(t-1), \mathbf{H}) - d_j(\mathbf{G}(t-1))c \tag{7}$$

Note that the notion of an improving path allows for myopic behavior on part of the players. The active-passive tuple of players in any stage do not take into account the consequences of their actions for the

[8] Even though the number of rounds is open-ended, we will see that a pws-equilibrium will be reached in a *finite* number of rounds.

[9] Within each round we can specify that the active player first deletes links and then proposes a new link. We will see that in our framework this specification of order of play is inconsequential.

[10] In Jackson and Watts (2002), such a path is called a *simultaneous* improving path, to refer to the fact that a change in the network entails both deletion and addition of links. For the sake of brevity we simply refer to it as an improving path.

players that will follow later.[11]

A set $\mathcal{G}$ of networks is a *cycle* if for any $\mathbf{G}, \mathbf{G}' \in \mathcal{G}$, there exists an improving path from $\mathbf{G}$ to $\mathbf{G}'$. A cycle is *maximal* if it is not a proper subset of a cycle. Finally, a cycle $\mathcal{G}$ is *closed* if no element of $\mathcal{G}$ is on an improving path to a network $\mathbf{G}' \notin \mathcal{G}$. A simple adaptation of Jackson and Watts (2002, Lemma 1) establishes that an improving path leads to a pws-equilibrium or a closed cycle.

3 The strategic complementarities model

We consider a model where actions of players within a network are strategic complements; further, each player i's actions x_i and y_i across the two networks are also strategic complements. Hence the utility of player i is differentially affected from links across the two different networks. Following Ballester et al. (2006), we formally capture this using a linear-quadratic utility function:

$$u_i(\mathbf{a}_i, \mathbf{a}_{-i}, \mathbf{G}, \mathbf{H}) = \left[x_i - \frac{1}{2}x_i^2\right] + \left[y_i - \frac{1}{2}y_i^2\right] + \left[\lambda x_i \sum_{j\in\mathcal{N}} g_{ij}x_j\right] + \left[\psi y_i \sum_{k\in\mathcal{N}} h_{ik}y_k\right] + \gamma x_i y_i \tag{8}$$

The first two terms in brackets capture concavity in own effort in networks $\mathbf{G}$ and $\mathbf{H}$ respectively where the negative quadratic term is the cost of actions in the two networks. We assume that the parameters satisfy $\lambda > 0$ and $\psi > 0$ so that the links within each network are strategic complements as shown in the next two terms. However, to remain agnostic *a priori* about link formation in $\mathbf{G}$, we will assume that λ is sufficiently small. By ensuring that strategic complementarities within $\mathbf{G}$ are sufficiently weak, it permits us to clearly draw out the role of the topology of $\mathbf{H}$ on the network formation process in $\mathbf{G}$.[12] The last term is crucial in that it captures the strategic complementarity between actions in the two networks with $\gamma \in (0,1)$ capturing the strength of this complementarity. Thus, once the two networks are in place, players are engaged in a non-cooperative game on the two networks and simultaneously choose their action vector. An example of this model is provided below. Additional applications are presented in Section 6.

Example 3.1: Consider N firms that are local monopolies producing goods x and y that are complements in consumption with demand functions respectively:

$$p_i^x = 1 - \frac{1}{2}x_i + \frac{\gamma}{2}y_i, \quad p_i^y = 1 - \frac{1}{2}y_i + \frac{\gamma}{2}x_i$$

Firms can enter into R & D collaborative alliance with other firms for the production of good x (given by the network $\mathbf{G}$) as well as good y (given by the network $\mathbf{H}$). Collaboration alliances reduce the (constant

[11] Such a formulation is fairly standard (Jackson and Watts 2002, p. 273) and allows tractability as compared to a far-sighted network formation approach (Dutta et al. 2005). It would hold for example when players highly discount the future.

[12] This assumption also serves an additional technical purpose. It simplifies the mathematics by allowing us to ignore terms involving λ of order 2 and higher. We argue later that our results continue to hold even if terms involving λ of order 2 and higher are taken into account in our calculations.

with respect to output) marginal costs of production for each firm according to:

$$c_i^x = c_0^x - \lambda \sum_{j \in \mathcal{N}} g_{ij} x_j, \quad c_i^y = c_0^y - \psi \sum_{k \in \mathcal{N}} h_{ik} y_k, \quad 0 < c_0^x, c_0^y < 1$$

Then profits for each firm are of the form given by (8). ■

We start by characterizing the Nash equilibrium in actions. For (8) the first order conditions for $i \in \mathcal{N}$ are given by:

$$\begin{aligned} \frac{\partial u_i}{\partial x_i} &= 1 - x_i + \lambda \sum_{j=1}^{N} g_{ik} x_j + \gamma y_i = 0 \\ \frac{\partial u_i}{\partial y_i} &= 1 - y_i + \psi \sum_{k=1}^{N} h_{ik} y_k + \gamma x_i = 0 \end{aligned}$$

Clearly $(x_i, y_i) = (0, 0)$ for all $i \in \mathcal{N}$ is not a Nash equilibrium. Writing the pair of first order conditions in matrix form, we get:

$$\mathbf{x} = \mathbf{1} + \lambda \mathbf{G} \mathbf{x} + \gamma \mathbf{y} \tag{9}$$

$$\mathbf{y} = \mathbf{1} + \psi \mathbf{H} \mathbf{y} + \gamma \mathbf{x} \tag{10}$$

Assuming $\psi \mu_{\max}(\mathbf{H}) < 1$, it follows that $\mathbf{M}(\mathbf{H},\psi) \equiv [m_{ij}]$=$[\mathbf{I} - \psi \mathbf{H}]^{-1}$ exists and is non-negative. From (10) we then have:

$$\mathbf{y} = \mathbf{M}(\mathbf{H},\psi)\mathbf{1} + \gamma \mathbf{M}(\mathbf{H},\psi)\mathbf{x} = \mathbf{b}(\mathbf{H},\psi,\mathbf{1}) + \gamma \mathbf{M}(\mathbf{H},\psi)\mathbf{x} \tag{11}$$

Substituting into (9):

$$\left[\mathbf{I} - \lambda \mathbf{G} - \gamma^2 \mathbf{M}(\mathbf{H},\psi)\right] \mathbf{x} = \mathbf{1} + \gamma \mathbf{b}(\mathbf{H},\psi,\mathbf{1})$$

To capture the interdependencies across networks, we introduce the notion of a composite network $\mathbf{L}$. Let $\mathbf{L} \equiv [l_{ij}] = \lambda \mathbf{G} + \gamma^2 \mathbf{M}(\mathbf{H},\psi)$ and $\boldsymbol{\alpha}_{\mathbf{H}} = \gamma \mathbf{b}(\mathbf{H},\psi,\mathbf{1}) + \mathbf{1}$. Assuming $\mu_{\max}(\mathbf{L}) < 1$,[13] then:

$$\mathbf{x}^*(\mathbf{G},\mathbf{H}) = [\mathbf{I} - \mathbf{L}]^{-1} \boldsymbol{\alpha}_{\mathbf{H}} = \mathbf{b}(\mathbf{L}, \boldsymbol{\alpha}_{\mathbf{H}}) \tag{12}$$

Substituting into (11) and letting $\boldsymbol{\beta}_{\mathbf{L}} = \gamma \mathbf{b}(\mathbf{L}, \boldsymbol{\alpha}_{\mathbf{H}}) + \mathbf{1}$:

$$\mathbf{y}^*(\mathbf{G},\mathbf{H}) = \mathbf{M}(\mathbf{H},\psi)(\mathbf{1} + \gamma \mathbf{b}(\mathbf{L}, \boldsymbol{\alpha}_{\mathbf{H}})) = \mathbf{b}(\mathbf{H},\psi,\boldsymbol{\beta}_{\mathbf{L}}) \tag{13}$$

Therefore the Nash equilibrium level of action $x_i^*(\mathbf{G},\mathbf{H})$ is determined by player i's $\boldsymbol{\alpha}_{\mathbf{H}}$-weighted KB centrality in network $\mathbf{L}$, while $y_i^*(\mathbf{G},\mathbf{H})$ is determined by player i's $\boldsymbol{\beta}_{\mathbf{L}}$-weighted KB centrality in network

[13] This implicitly places restrictions on the parameters λ, ψ, and γ such that the feedback loops in each network are bounded and players' actions converge to a Nash equilibrium. In particular, a small value of γ implies weak complementarity in actions across networks. Also, note that given the nature of dependencies across networks, especially involving the KB centralities, it is not possible to focus on degrees of the players for solving the problem.

$\mathbf{H}$. We can now compute the reduced form utilities of the players. Using the first order conditions it follows that:

$$U_i(\mathbf{G},\mathbf{H}) = \frac{1}{2}\left[y_i^*(\mathbf{G},\mathbf{H}) - x_i^*(\mathbf{G},\mathbf{H})\right]^2 + (1-\gamma)x_i^*(\mathbf{G},\mathbf{H})y_i^*(\mathbf{G},\mathbf{H}) \tag{14}$$

To simplify notation we write the Nash equilibrium actions as x_i^* and y_i^* and drop the reference to $\mathbf{G}$ and $\mathbf{H}$ when there is no ambiguity. We begin by considering the incremental utility of players from link formation.

Lemma 1 *Consider a network* $\mathbf{G}$*. If* $b_j(\mathbf{H},\psi,\mathbf{1}) \geq b_k(\mathbf{H},\psi,\mathbf{1})$*, and* $\mathbf{N}_k(\mathbf{G}) \subseteq \mathbf{N}_j(\mathbf{G})$*, then:*

$$U_i(\mathbf{G}+ij,\mathbf{H}) - U_i(\mathbf{G},\mathbf{H}) \geq U_i(\mathbf{G}+ik,\mathbf{H}) - U_i(\mathbf{G},\mathbf{H}) \tag{15}$$

$$U_j(\mathbf{G}+ij,\mathbf{H}) - U_j(\mathbf{G},\mathbf{H}) \geq U_k(\mathbf{G}+ik,\mathbf{H}) - U_k(\mathbf{G},\mathbf{H}) \tag{16}$$

$$U_i(\mathbf{G}+ij,\mathbf{H}) - U_i(\mathbf{G},\mathbf{H}) > U_i(\mathbf{G},\mathbf{H}) - U_i(\mathbf{G}-ik,\mathbf{H}) \tag{17}$$

$$U_j(\mathbf{G}+ij,\mathbf{H}) - U_j(\mathbf{G},\mathbf{H}) > U_k(\mathbf{G},\mathbf{H}) - U_k(\mathbf{G}-ik,\mathbf{H}) \tag{18}$$

The inequality in (15) and (16) is strict if $b_j(\mathbf{H},\psi,\mathbf{1}) > b_k(\mathbf{H},\psi,\mathbf{1})$ *and/or* $\mathbf{N}_k(\mathbf{G}) \subset \mathbf{N}_j(\mathbf{G})$*.*

Inequalities (15) and (17) examine the incremental utility of the same player from linking with partners having different KB centralities in $\mathbf{H}$. Specifically (15) keeps the base network $\mathbf{G}$ fixed and shows that player i gains more from linking with player j than with player k, if j is more central in $\mathbf{H}$ than k, and j's neighborhood in $\mathbf{G}$ contains that of k. Inequality (17) is a restricted "increasing returns" property that allows the base network to change. It states that incremental utility is increasing in links if the succeeding link is with a partner who is more central in $\mathbf{H}$, and whose neighborhood in $\mathbf{G}$ subsumes that of the predecessor. Analogously, inequalities (16) and (18) compare the incremental utility of a more central player j in $\mathbf{H}$ to that of a less central player k in $\mathbf{H}$, when both form a link with the same partner. Inequality (16) shows that j's incremental utility from adding the link ij to the network $\mathbf{G}$ is greater than that of k from adding the link ik to $\mathbf{G}$. Inequality (18) establishes the same claim while allowing the base network to change.

Given networks $\mathbf{G}$ and $\mathbf{G}'$, we say that $\mathbf{G}'$ is *denser* than $\mathbf{G}$, represented as $\mathbf{G} \subset \mathbf{G}'$, if $ij \in \mathbf{G}$ implies $ij \in \mathbf{G}'$, and there exists some $kl \in \mathbf{G}'$ such that $kl \notin \mathbf{G}$. The next result shows that incremental utility to a player from a link is greater in a denser network since the feedback effects are stronger. This result in turn establishes the existence of a pws-equilibrium network.

Lemma 2 *Suppose* $\mathbf{G} \subset \mathbf{G}'$ *and* $ij \notin \mathbf{G}'$*. Then:*

$$U_i(\mathbf{G}'+ij,\mathbf{H}) - U_i(\mathbf{G}',\mathbf{H}) > U_i(\mathbf{G}+ij,\mathbf{H}) - U_i(\mathbf{G},\mathbf{H}) > 0 \tag{19}$$

In particular, a pws-equilibrium exists.

Lemma 2 implies that players do not have an incentive to delete existing links along an improving path. However, if existing links are maintained, then it is not possible to have a closed cycle on an improving path. It follows from Jackson and Watts (2002, Lemma 1) that an improving path will lead to a pws-equilibrium network. Note that the networks on an improving path between $\mathbf{G}(0) \equiv \mathbf{G}^e$ and the pws-equilibrium will be called *interim* networks.

Our next result will play an important role in the characterization of pws-equilibrium networks. It states that if player i has a mutually profitable link with a partner k in network $\mathbf{G}$, then i has a mutually profitable link with any player who is at least as central as k in the network $\mathbf{H}$, and whose neighborhood in $\mathbf{G}$ subsumes that of k.

Lemma 3 *Suppose players k and i have an incentive to form a link in the network $\mathbf{G}$. If $b_j(\mathbf{H}, \psi, \mathbf{1}) \geq b_k(\mathbf{H}, \psi, \mathbf{1})$ and $\mathbf{N}_k(\mathbf{G}) \subseteq \mathbf{N}_j(\mathbf{G})$, then players j and i have an incentive to form a link in the network $\mathbf{G}'$, where $\mathbf{G} \subseteq \mathbf{G}'$.*

4 Characterization of pws-equilibrium

We will first consider the case where $\mathbf{H}$ is a NSG and examine the implications of the topology of $\mathbf{H}$ for equilibrium network formation in $\mathbf{G}$. We will then consider the case where $\mathbf{H}$ is a regular graph. Throughout this section we will assume that the attenuation parameter λ is sufficiently small.

4.1 Network H is a NSG

Recall that players are indexed according to their degree in $\mathbf{H}$. To simplify the exposition we will assume there are no isolated players in $\mathbf{H}$. From Lemma 2, the improving path from $\mathbf{G}(0) \equiv \mathbf{G}^e$ leads to a pws-equilibrium network $\mathbf{G}$. Our objective is to characterize $\mathbf{G}$. We will let $\mathcal{P}(\mathbf{G}) = \{P_0(\mathbf{G}), P_1(\mathbf{G}), ..., P_n(\mathbf{G})\}$ denote the degree partition of $\mathbf{G}$.

Consider round 1 and recall that active players move in the order of their index. Consider the active player $1 \in P_1(\mathbf{H})$ in round 1. Since the process starts at $\mathbf{G}^e$, there are no links to delete. From (15), the highest incremental utility accrues from a link with player $N \in P_m(\mathbf{H})$. Player 1 will propose the link $1N$ if the incremental utility exceeds the cost of link formation, c, i.e. $U_1(\mathbf{G}^e + 1N, \mathbf{H}) - U_1(\mathbf{G}^e, \mathbf{H}) > c$. To simplify the exposition we will assume that this condition holds. Since $b_N(\mathbf{H}, \psi, \mathbf{1}) > b_1(\mathbf{H}, \psi, \mathbf{1})$ and $\mathbf{N}_N(\mathbf{G}^e) = \mathbf{N}_1(\mathbf{G}^e) = \emptyset$, it follows from (16) that player N will reciprocate the link. Therefore the improving path from $\mathbf{G}^e$ leads to $\mathbf{G}(1) \equiv \mathbf{G}^e + 1N$. We now claim that each active player $k \in \{2, 3, ..., N-1\}$ in round 1 will propose a link to player N and player N will reciprocate. Suppose this is true for player $k \in \{2, 3, ..., N-2\}$ leading to the network $\mathbf{G}(k)$. We show that this also true for active player $(k+1)$. Since $b_{k+1}(\mathbf{H}, \psi, \mathbf{1}) \geq b_k(\mathbf{H}, \psi, \mathbf{1})$ and $\mathbf{N}_k(\mathbf{G}(k-1)) = \mathbf{N}_{k+1}(\mathbf{G}(k-1)) = \emptyset$, it follows from Lemma 3 that player

$(k+1)$ can profitably link with player N. Since $b_N(\mathbf{H},\psi,\mathbf{1}) \geq b_i(\mathbf{H},\psi,\mathbf{1})$ and $\mathbf{N}_i(\mathbf{G}(k)) \subset \mathbf{N}_N(\mathbf{G}(k))$ $\forall i \in \mathcal{N}\backslash\{N\}$, it follows from (15) that the most profitable link for player $k+1$ is with player N:

$$U_{k+1}(\mathbf{G}(k)+(k+1)N,\mathbf{H}) - U_{k+1}(\mathbf{G}(k),\mathbf{H}) > U_{k+1}(\mathbf{G}(k)+(k+1)i,\mathbf{H}) - U_{k+1}(\mathbf{G}(k),\mathbf{H}), \quad \forall i \in \mathcal{N}\backslash\{N\}$$

Noting that $\mathbf{G}(k) = \mathbf{G}(k-1)+kN$, it follows from (17) that player N will reciprocate:

$$U_N(\mathbf{G}(k)+(k+1)N,\mathbf{H}) - U_N(\mathbf{G}(k),\mathbf{H}) > U_N(\mathbf{G}(k),\mathbf{H}) - U_N(\mathbf{G}(k)-kN,\mathbf{H}) > c$$

This establishes the induction result. Therefore, at the end of stage $N-1$, $|\mathbf{N}_N(\mathbf{G}(N-1))| = N-1$, i.e. player N is linked to all $N-1$ players. In stage N, it follows from (19) that the active player N will not delete any links and has no additional links to propose. Therefore player $N \in P_n(\mathbf{G})$. Now suppose that $|P_m(\mathbf{H})| = l > 1$ so that $P_m(\mathbf{H}) = \{N-l+1, N-l+2, ..., N\}$. Then the same argument applies to each round $r \in \{2,3,...,l\}$ in which each active player $j = 1,2,..,N-r$ proposes a link with player $N-r+1 \in P_m(\mathbf{H})$ and does not delete existing links. Player $N-r+1$ reciprocates the offer and therefore $\mathbf{N}_i(\mathbf{G}) = N-1$ $\forall i \in P_m(\mathbf{H})$. This establishes the following result:

Proposition 1 *Suppose* $U_1(\mathbf{G}^e+1N,\mathbf{H}) - U_1(\mathbf{G}^e,\mathbf{H}) > c$*,* $\mathbf{H}$ *is a NSG, and the improving path from* $\mathbf{G}(0) \equiv \mathbf{G}^e$ *leads to the network* $\mathbf{G}$*. Then* $P_m(\mathbf{H}) \subseteq P_n(\mathbf{G})$ *and each player in* $P_m(\mathbf{H})$ *is linked to all players in the network* $\mathbf{G}$*.*

From the argument above we see that interim networks along an improving path from $\mathbf{G}(0) \equiv \mathbf{G}^e$ evolve in a very specific way. Suppose $\mathbf{G}'$ is the interim network at the end of a given round when all N players have moved in the order of their index. A player who is more central in $\mathbf{H}$, say i, will have a neighborhood in $\mathbf{G}'$ that nests the neighborhood of a less central player in $\mathbf{H}$, say j. This is because the mutual profitability of the link jk during a given round ensures the mutual profitability of the link ik as well. This observation leads to a simplification of Lemma 3. To ascertain whether two players have a mutually profitable link in an interim network on an improving path, it suffices to check the centrality of the potential partners in $\mathbf{H}$.

Lemma 4 *Suppose players* k *and* i *have an incentive to form a link in an interim network* $\mathbf{G}'$ *on an improving path. If* $b_j(\mathbf{H},\psi,\mathbf{1}) \geq b_k(\mathbf{H},\psi,\mathbf{1})$*, then players* j *and* i *have an incentive to form a link in an interim network* $\mathbf{G}''$*, where* $\mathbf{G}' \subseteq \mathbf{G}''$*.*

Armed with this result we now consider the players in $P_1(\mathbf{G})$ or those with the fewest connections in the pws-equilibrium network $\mathbf{G}$. We will first verify that the least central players in $\mathbf{H}$ will have the fewest connections in $\mathbf{G}$. This is easy to demonstrate. Suppose instead that player $i \in P_1(\mathbf{H}) \cap P_l(\mathbf{G})$ for some $l = 2,3,...,n$. Then i has strictly greater degree than a player, say j, in $P_1(\mathbf{G})$. However $b_j(\mathbf{H},\psi,1) \geq b_i(\mathbf{H},\psi,1)$, and therefore any mutually profitable link along an improving path for player i will be mutually profitable for player j as well by virtue of Lemma 4. Therefore $d_j(\mathbf{G}) \geq d_i(\mathbf{G})$ which contradicts $j \in P_1(\mathbf{G})$. Therefore the set of players with the fewest connections in $\mathbf{G}$ includes the least central players in $\mathbf{H}$. It can also be established that each player $i \in P_1(\mathbf{G})$ can only have links with players in $P_n(\mathbf{G})$ and no others yielding the following characterization.

Proposition 2 *Suppose* $\mathbf{H}$ *is a NSG and the improving path from* $\mathbf{G}(0) \equiv \mathbf{G}^e$ *leads to the network* $\mathbf{G}$*. Then* $P_1(\mathbf{H}) \subseteq P_1(\mathbf{G})$ *and* $\mathbf{N}_i(\mathbf{G}) = P_n(\mathbf{G})$ $\forall i \in P_1(\mathbf{G})$*.*

At this point we digress briefly to consider the extreme cases of empty and complete networks as candidates for a pws-equilibrium. It is immediately obvious that for sufficiently low (respectively, high) costs of link formation the pws-equilibrium network will be complete (respectively, empty). Recalling the indexation of the players, suppose that the linking cost c satisfies:

$$0 \leq c < U_1(\mathbf{G}^e + 12, \mathbf{H}) - U_1(\mathbf{G}^e, \mathbf{H})$$

i.e. the least central player in $\mathbf{H}$ finds it profitable to forge a link with the second least central player in $\mathbf{H}$ when no connections exist among the players in the $\mathbf{G}$ network. Since the RHS above is the minimum of incremental utilities accruing to players along an improving path, and this minimum exceeds the cost of link formation, it follows from Lemma 4 that each player can form a mutually profitable link with the remaining $N-1$ players. Therefore, this is a sufficient condition under which the improving path will lead to the complete network $\mathbf{G}^c$.

Next we provide a sufficient condition for the empty network. Suppose that:

$$U_{N-1}(\mathbf{G}^e + (N-1)N, \mathbf{H}) - U_{N-1}(\mathbf{G}^e, \mathbf{H}) \leq c$$

i.e. at least one of the two most central players in H has no incentive to form a link in the empty network. It follows that each active player from stage 1 onwards will not have an incentive to announce a link with a partner since no link is mutually profitable. Thus the improving path leads to the empty network. Of course there is a potential coordination problem here. It is quite possible that in some non-empty (denser) network the incremental utilities for some players exceed c and consequently mutually profitable links are possible. However, since players respond myopically to the existing network, the improving path remains "trapped" at $\mathbf{G}^e$.

We now turn to those players who are neither the most central or the least central in network $\mathbf{H}$ and occupy the intermediate sets in the degree partition of $\mathbf{G}$. It turns out that sets in the degree partition of a pws-equilibrium $\mathbf{G}$ cannot partially overlap with those in the degree partition of $\mathbf{H}$, i.e. it is not possible that both $P_r(\mathbf{H}) \cap P_l(\mathbf{G}) \neq \emptyset$ and $P_r(\mathbf{H}) \cap P_{l'}(\mathbf{G}) \neq \emptyset$ for $l \neq l'$. A set in $\mathcal{P}(\mathbf{G})$ either coincides with an element of $\mathcal{P}(\mathbf{H})$ or is the union of successive elements of $\mathcal{P}(\mathbf{H})$. Putting it simply, two players who are equally central in $\mathbf{H}$ cannot have different degrees in $\mathbf{G}$. However, it is possible that two players with different centralities in $\mathbf{H}$ have the same degree in $\mathbf{G}$.

Proposition 3 (*Non-Partial Overlap Property*) *Suppose* $\mathbf{H}$ *is a NSG and the improving path from* $\mathbf{G}(0) \equiv \mathbf{G}^e$ *leads to the network* $\mathbf{G}$*. The degree partition* $\mathcal{P}(\mathbf{G})$ *displays the following property: For each* $l = 1, 2, ..., n-1$*, let* $R_l = [r', r'+1, ..., r'']$ *be the set of indices such that* $P_r(\mathbf{H}) \cap P_l(\mathbf{G}) \neq \emptyset$ *for* $r \in R_l$ *and* $P_r(\mathbf{H}) \cap P_l(\mathbf{G}) = \emptyset$ *otherwise. Then* $P_l(\mathbf{G}) = \cup_{r \in R_l} P_r(\mathbf{H})$*.*

The non-partial overlap property can be described as an "equal treatment in equilibrium" property: Two players with the same Katz-Bonacich centrality in network $\mathbf{H}$ cannot be treated differently in the equilibrium network $\mathbf{G}$. It is possible however for two players with different Katz-Bonacich centralities in $\mathbf{H}$ to be treated the same in the equilibrium network $\mathbf{G}$. As an illustration, consider the NSG in figure 4. Any set of players who have the same degree in $\mathbf{H}$, for instance players 1 and 2, also have the same degree in $\mathbf{G}$. This reveals two different effects in equilibrium. On one hand, we find that those with the highest (respectively, lowest) Bonacich centrality in $\mathbf{H}$ will have the highest (respectively, lowest) Bonacich centrality in $\mathbf{G}$. Hence we see a kind of "silver spoon effect" in equilibrium. Essentially, it says that preferential attachment is a wide-spread phenomenon. As long as there are strategic complementarities, nodes with high centrality in one network will continue to to be highly central in the other network as well.

On the other hand, we also see a "silver lining". Since $\mathbf{G}$ inherits a "coarser" degree partition from $\mathbf{H}$, there is clearly the possibility of limited mobility, i.e. players in certain degree partitions in $\mathbf{H}$ can be in the same or higher degree partitions of $\mathbf{G}$. This is largely driven by costs of link formation in $\mathbf{G}$ and we focus more on this issue of inequality in networks in the next section.

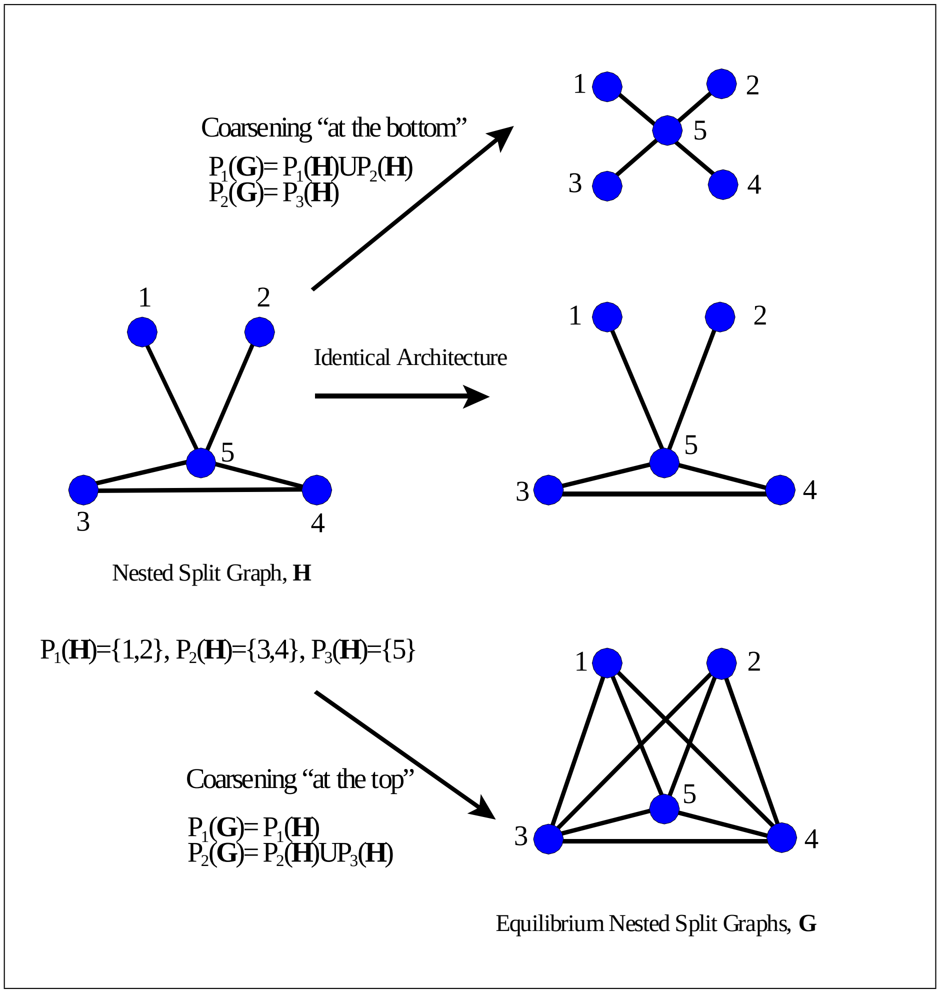

Figure 4: Non-Partial Overlap Property

Given that **H** is a NSG, there is another very useful and by now obvious practical consequence. As shown by König et al. (2014), in a NSG the Bonacich centrality coincides with degree centrality. So the above statements about the silver spoon effect and limited mobility can be based on node degrees. A second important implication of the above proposition is that the cardinality of the degree partition of **G** cannot exceed that of **H**. In other words, **G** inherits a "coarser" degree partition from **H**. Considering figure 4, we can have a coarsening "at the bottom" when the lower elements in the degree partition of **H** are merged in **G**, or a coarsening "at the top" when the upper elements in the degree partition of **H** are combined in **G**. We now show that **G** will in fact inherit a NSG architecture.[14]

[14]Interestingly Avner Greif's (1994) study of individualist (Genoese) and collectivist (Maghribis) societies illustrates what can happen if such strategic complementarities across networks do not exist. Genoese traders who exemplified an individualistic society relied on contract law. Maghribi traders on the other hand were a part of a community with pre-existing ties and relied on their social ties for information sharing and multilateral punishments to support a collectivist equilibrium in their business network. The recorded history of wealth dynamics is non-existent in the case of the Maghribis, but the Genoese sources record wealth dynamics. These wealth dynamics illustrate how a society evolves when it is not driven by strategic

Proposition 4 *Suppose* $\mathbf{H}$ *is a NSG and the improving path from* $\mathbf{G}(0) \equiv \mathbf{G}^e$ *leads to the network* $\mathbf{G}$*. Then a pws-equilibrium* $\mathbf{G}$ *is empty, complete or has at most one non-singleton NSG component.*

The result follows by virtue of the non-partial overlap property. Intuitively, since the degree partition of $\mathbf{G}$ is composed either of individual elements of $\mathcal{P}(\mathbf{H})$ or union of elements of $\mathcal{P}(\mathbf{H})$, it follows that $\mathcal{P}(\mathbf{G})$ exhibits the nested neighborhood structure that is characteristic of a NSG. If there is more than one non-singleton component, then from each component we can pick the player who is most central in $\mathbf{H}$. The two players picked in this manner have a mutually profitable link from Lemma 4.[15]

The non-partial overlap property is also of practical significance in many instances because it reduces the number of architectures that have to considered as candidates for equilibrium. For example, if $\mathbf{H}$ is a star or an "interlinked star" as in figure 5, then there is no NSG with a coarser degree partition that respects the non-partial overlap property. Therefore, if a pws equilibrium $\mathbf{G}$ other than the empty or complete exists for some intermediate level of linking costs, then $\mathbf{G}$ must have the same architecture as $\mathbf{H}$.

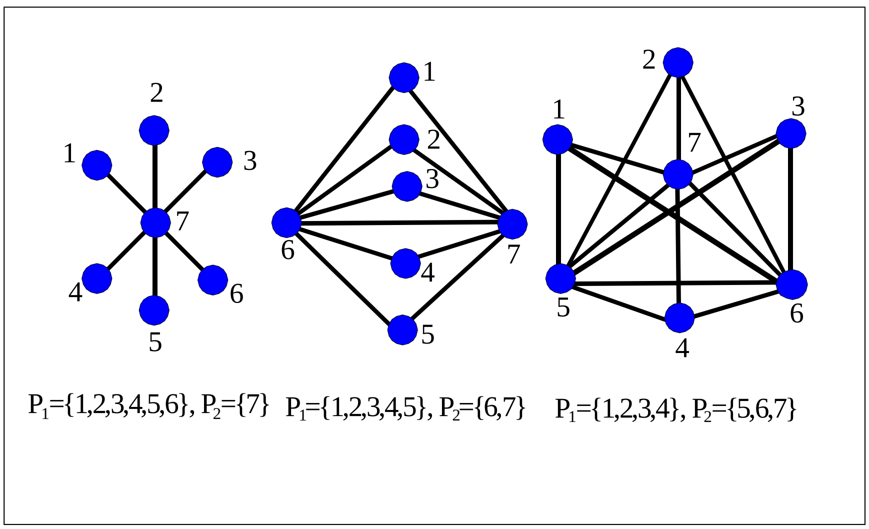

Figure 5: Star and "Interlinked" Stars Networks

Remark: Our assumption that λ is sufficiently small, so that λ^2 and higher powers can be ignored, simplified the mathematical expressions involved. In particular, reduced utility did not involve higher

complementarities across networks. Initially "*...trade was concentrated, on the whole, in the hands of a few noble families, and less than* 10 *percent of the merchants invested* 70 *percent of the total. In the cartulary of Oberto Scriba* (1186), 10 *percent of the families invested less than* 60 *percent*". This can be contrasted with the year 1376, when "*commoners who paid customs in Genoa exceeded nobles* (295 vs. 279), *and the share of the latter amounted to only* 64 *percent of the total* (Kedar 1976, pp.51-52)" (Greif, 1994).

[15]It is also worth pointing out that König et al. (2014) study network formation under strategic complementarities in a single network setting and find that the equilibrium architecture is a NSG. While NSG seems to be intuitive in this context, they can also arise under strategic substitutes. This is shown in a recent paper by Kinateder and Merlino (2017) who study public goods on networks under heterogeneous costs and values.

powers of the $\mathbf{G}$ matrix. We observe however that the network $\mathbf{G}$ evolves along an improving path with "preferential attachment" to more central players in $\mathbf{H}$. At any given stage of the network formation process, the interim networks on the improving path have the configuration of a NSG. Therefore, relaxing the restriction that λ be sufficiently small (subject of course to the condition that $\mu_{\max}(\mathbf{L}) < 1$ to bound the feedback loops) and incorporating higher powers of the $\mathbf{G}$ matrix into the calculation will not affect our results.

4.2 Network H is a KB-Regular Graph

We will now assume that $\mathbf{H}$ is a KB-regular graph in which all players are equally central. We will show that the pws-equilibrium network $\mathbf{G}$ is regular as well. In fact, depending on costs of link formation, $\mathbf{G}$ is either empty or complete.

Let $\mathbf{H}$ be a KB-regular graph. Consider round 1 and the active player $1 \in P_1(\mathbf{H})$ in stage 1. Without loss of generality, we can focus on player N, since all players are equally central in $\mathbf{H}$. If $U_1(\mathbf{G}^e + 1N, \mathbf{H}) - U_1(\mathbf{G}^e, \mathbf{H}) \leq c$, then no link will be proposed by player 1. Since all players are ex-ante identical, no links are formed subsequently and the improving path ends in $\mathbf{G}^e$. If $U_1(\mathbf{G}^e + 1N, \mathbf{H}) - U_1(\mathbf{G}^e, \mathbf{H}) > c$, then the link $1N$ is formed. Subsequently each active player $k \in \{2, 3, ..., N-1\}$ in round 1 will propose a link to player N and player N will reciprocate. Therefore player N is maximally connected at the end of round 1. From Lemma 2, each subsequent round ends with a player getting maximally connected. The improving path therefore leads to the complete network. We have therefore proved the following result.

Proposition 5 *Suppose* $\mathbf{H}$ *is a KB-regular graph and the improving path from* $\mathbf{G}(0) \equiv \mathbf{G}^e$ *leads to the network* $\mathbf{G}$*. Then* $\mathbf{G}$ *is either empty or complete.*

5 Network Inequality

We have seen that when $\mathbf{H}$ is a NSG, then the network $\mathbf{G}$ assumes a NSG architecture with a possibly coarser degree partition. An important question then is: How does the process of endogenous link formation influence the degree distribution in $\mathbf{G}$ relative to that in $\mathbf{H}$? We will see that the degree distribution in $\mathbf{G}$ is strongly impacted by the cost of link formation in $\mathbf{G}$. To show this result we will fix a reference network and establish a set of baseline incremental utilities with respect to this network.

Let $\mathbf{G}^*$ denote the network that is identical to $\mathbf{H}$. Recalling that $\mathbf{H}$ (and therefore $\mathbf{G}^*$) has the degree partition $\mathcal{P}(\mathbf{H}) = \{P_0(\mathbf{H}),\ P_1(\mathbf{H}), ..., P_m(\mathbf{H})\}$, define the following incremental utilities:

$$\begin{aligned}
\underline{\delta}_l &= U_i(\mathbf{G}^* + ik, \mathbf{H}) - U_i(\mathbf{G}^*, \mathbf{H}), \quad i \in P_l(\mathbf{H}),\ k \in P_{m-l}(\mathbf{H}),\ l = 1, 2, ..., \lfloor m/2 \rfloor - 1 \\
\overline{\delta}_l &= U_i(\mathbf{G}^*, \mathbf{H}) - U_i(\mathbf{G}^* - ij, \mathbf{H}), \quad i \in P_l(\mathbf{H}),\ j \in P_{m-l+1}(\mathbf{H}),\ l = 1, 2, ..., \lfloor m/2 \rfloor
\end{aligned}$$

Note that $\underline{\delta}_l$ and $\overline{\delta}_l$ depend only on the relevant elements of the partition $\mathcal{P}(\mathbf{H})$ and not on the identity of the players who comprise these partitions. Let:

$$\underline{\delta} = \max\{\underline{\delta}_l, l = 1, 2, ..., \lfloor m/2 \rfloor - 1\}, \quad \overline{\delta} = \min\{\overline{\delta}_l, l = 1, 2, ..., \lfloor m/2 \rfloor\}$$

If $\underline{\delta} < \overline{\delta}$, then cost of link formation $c \in (\underline{\delta}, \overline{\delta})$ can support $\mathbf{G}^*$ as a pws-equilibrium network. For this case:

$$\underline{\delta}_l < c, \quad l = 1, 2, ..., \lfloor m/2 \rfloor - 1 \tag{20}$$

$$c < \overline{\delta}_l, \quad l = 1, 2, ..., \lfloor m/2 \rfloor \tag{21}$$

The first condition (20) ensures that a player in $P_l(\mathbf{G}^*)$ has no incentive to propose a link to a player in $P_{m-l}(\mathbf{G}^*)$. From (15) in Lemma 1 it follows that players in $P_l(\mathbf{G}^*)$ have no incentive to propose links to less central players in $P_{m-l-1}(\mathbf{G}^*), P_{m-l-2}(\mathbf{G}^*), ..., P_{l+1}(\mathbf{G}^*)$ etc., either. The second condition (21) ensures that a player in $P_l(\mathbf{G}^*)$ has no incentive to delete links with the least central of its partners in $\mathbf{H}$. It then follows from (15) that players in $P_l(\mathbf{G}^*)$ have no incentive to delete any existing links. Therefore, for $c \in (\underline{\delta}, \overline{\delta})$, there is a pws-equilibrium network that mirrors the degree distribution of $\mathbf{H}$.[16]

Before proceeding, we need to establish a measure that can compare the variation in degree distribution across two networks. There are numerous ways in which inequality in degree can be measured, and here we choose a particularly simple measure that looks at “first degree dominance" in the degree distribution of the players. We will say that the degree distribution of $\mathbf{G}$ *first order dominates* the degree distribution of $\mathbf{G}'$ if $d_i(\mathbf{G}) \geq d_i(\mathbf{G}')$, $\forall i \in \mathcal{N}$ with strict inequality for at least one player i.

Proposition 6 *Suppose $\mathbf{H}$ is a regular network and the improving path from $\mathbf{G}(0) \equiv \mathbf{G}^e$ leads to some non-empty network $\mathbf{G}$. If $\overline{\delta} \leq \overline{\delta}_1 < c$, then the degree distribution of $\mathbf{G}^*$ first order dominates that of $\mathbf{G}$. If $0 < c < \underline{\delta}_1 \leq \underline{\delta}$, and $\mathbf{G}^*$ lies on the improving path from $\mathbf{G}^e$, then the degree distribution of $\mathbf{G}$ first order dominates that of $\mathbf{G}^*$.*

To prove this result, suppose linking costs are high relative to the levels required to support $\mathbf{G}^*$ as a pws-equilibrium. In particular, suppose that $\overline{\delta} \leq \overline{\delta}_1 < c$. Note that $\mathbf{G}^*$ cannot be an interim network on the improving path leading to $\mathbf{G}$. Since $\mathbf{G}^*$ includes all links between minimally and maximally connected players, if $\mathbf{G}^*$ was on the improving path, then in some interim network $\mathbf{G}' \subset \mathbf{G}^*$ the active player $i \in P_1(\mathbf{H})$ would have formed a profitable link with $j \in P_m(\mathbf{H})$. However, by virtue of Lemma 2:

$$U_i(\mathbf{G}' + ij, \mathbf{H}) - U_i(\mathbf{G}', \mathbf{H}) < U_i(\mathbf{G}^* + ij, \mathbf{H}) - U_i(\mathbf{G}^*, \mathbf{H}) = \overline{\delta}_1 < c \tag{22}$$

contradicting the profitability of the link.

[16] This is not to say that there will always be an improving path leading to $\mathbf{G}^*$. Since players behave myopically along an improving path, it is possible that the process will lead to a pws-equilibrium network different from $\mathbf{G}^*$.

Next, note that the limiting network $\mathbf{G} \subset \mathbf{G}^*$. If this is not the case, then there is at least one link in $\mathbf{G}$ that does not exist in $\mathbf{G}^*$. Let kl be the *first* link that is formed along the improving path that does not exist in $\mathbf{G}^*$ and let $\mathbf{G}'$ be the interim network when kl is established. Then, by the choice of the link, $\mathbf{G}' \subseteq \mathbf{G}^*$. Now let $c' \in (\underline{\delta}, \overline{\delta})$ denote any cost supporting $\mathbf{G}^*$ as a pws-equilibrium and note that $c' < \overline{\delta} \leq \overline{\delta}_1 < c$. From Lemma 2:

$$U_k(\mathbf{G}^* + kl, \mathbf{H}) - U_k(\mathbf{G}^*, \mathbf{H}) \geq U_k(\mathbf{G}' + kl, \mathbf{H}) - U_k(\mathbf{G}', \mathbf{H}) > c > c'$$

and identically for player l. Therefore players kl have a mutually profitable link in $\mathbf{G}^*$ for $c' \in (\underline{\delta}, \overline{\delta})$ contradicting its pws-equilibrium property.

The limiting network $\mathbf{G}$ must be less dense than $\mathbf{G}^*$. Now applying the same reasoning as in (22), it follows that $\mathbf{G}$ does not have any links of the form ij, $i \in P_1(\mathbf{H})$ and $j \in P_m(\mathbf{H})$, i.e. links between the minimally and maximally connected players. From Lemma 1, equation (15), any player who has at least one link must be linked to players in $P_m(\mathbf{H})$. It follows that $d_i(\mathbf{G}) \leq d_i(\mathbf{G}^*)$ $\forall i \in \mathcal{N}$ and $d_i(\mathbf{G}) < d_i(\mathbf{G}^*)$ $\forall i \in P_1(\mathbf{H}) \cup P_m(\mathbf{H})$. The players at the lower extreme of the degree distribution of $\mathbf{G}$ see a strict decrease in their number of connections (in fact they are now isolated in $\mathbf{G}$) while the players at the upper extreme continue to be maximally connected (though to a smaller set of players). To the extent that players at the lower end of the degree distribution are left worse off than those at the upper end, the first order domination of $\mathbf{G}$ by $\mathbf{G}^*$ can be construed as a move towards greater inequality in degree. Therefore, an increase in linking costs above $\overline{\delta}_1$ increases the inequality in degree distribution relative to $\mathbf{G}^*$.

Now consider the case where $c < \underline{\delta}_1$. The assumption that $\mathbf{G}^*$ lies on the improving path leading to $\mathbf{G}$ is needed to address the coordination problems created by strategic complementarity. Since the initial interim networks are less dense than $\mathbf{G}^*$, they may preclude the formation of profitable links that exist in $\mathbf{G}^*$. At any interim network $\mathbf{G}' \supset \mathbf{G}^*$, the active player $i \in P_1(\mathbf{H})$ profits by announcing a link with $k \in P_{m-1}(\mathbf{H})$:

$$U_i(\mathbf{G}' + ik, \mathbf{H}) - U_i(\mathbf{G}', \mathbf{H}) > U_i(\mathbf{G}^* + ik, \mathbf{H}) - U_i(\mathbf{G}^*, \mathbf{H}) = \underline{\delta}_1 > c$$

From Lemma 1, equation (16), player k will accept. It follows that the limiting network $\mathbf{G}$ will contain all links of the form ik, $i \in P_1(\mathbf{H})$ and $k \in P_{m-1}(\mathbf{H})$. Therefore $d_i(\mathbf{G}) \geq d_i(\mathbf{G}^*)$ $\forall i \in \mathcal{N}$ and $d_i(\mathbf{G}) > d_i(\mathbf{G}^*)$ $\forall i \in P_1(\mathbf{H}) \cup P_{m-1}(\mathbf{H})$. To the extent that players at the lower end of the degree distribution are now better off, a reduction in linking costs below $\underline{\delta}_1$ can be interpreted as decreasing the inequality in degree distribution relative to $\mathbf{G}^*$.

6 Path Dependence and Multigraphs: Some Applications

The goal of this section is to briefly illustrate several applications of the framework outlined in the previous sections. Individual actions are embedded in a social structure. Sociologists typically assume a network to exist and model the behavior of individuals embedded in it, while economists typically focus on the formation of such networks. Our paper illustrates how actions within the embedded structure of an existing network influence the formation of an alternate social structure and actions in it. Thus our paper ties these two perspectives and provides insights about how complementarity impacts actions and other social structures.

6.1 Intergenerational Mobility

Numerous studies have shown that family background has a significant influence on intergenerational mobility (see for example Becker and Tomes (1979, 1986) and Corak (2013)). The effect of family goes far beyond simply inheriting the traits of successful parents. Children of successful parents not only receive better education but also typically enjoy higher social status. The pre-existing network formed by the older generation is inherited by the subsequent generation. Children born into privileged backgrounds will typically be more centrally located in a network than those who are less fortunate. This mirrors our model well where we posit that network $\mathbf{H}$ is the inherited from the past generation and $\mathbf{y}$ denotes actions in the social sphere of the family. Interestingly, Datcher Loury (2006) mentions that in the United States about half of jobs are obtained through the individual's social network. Her study also reveals that jobs that are found through the "prior generation male" relatives paid the highest wages. Theoretical models like the one proposed by Mortensen and Vishwanath (1994), illustrate the positive impact of contacts on wages.

Next, consider a society that has inherited a highly unequal network like a NSG. When the returns to education are low, strong complementarity between x and y (as measured by γ) can induce intergenerational stickiness and reduce mobility. Interestingly, there may be other reasons for such stickiness. Corak and Piraino (2010, 2011) and Bingley, Corak and Westergard-Nielson (2012) find a very strong transmission of economic status between generations at the higher levels of status even in nations where intergenerational mobility is relatively high. The authors attribute this phenomenon to intergenerational transmission of employers who tend to prefer employees coming from families with higher status. The evidence vindicates the prediction of our model since more central players in the inherited network $\mathbf{H}$ have greater incentives to form ties in $\mathbf{G}$.

6.2 Research Collaboration

Unlike the textbook version of atomistic agents, in practice, the typical firm is embedded in social as well as professional networks. In fact, this idea is well documented in the management literature and includes both formal and informal networks. In a seminal paper on strategic networks, Gulati et al. (2000) write "*An understanding of the consequences of the ubiquitous growth of strategic networks emphasizes that firms are more properly viewed as connected to each other in multiple networks of resource and other flows.*" In this literature, the network in which firms are embedded is an asset in itself. Therefore, a comprehensive view of how collaboration evolves among firms requires an understanding of the pre-existing networks.

There is also an empirical literature on how the position of a firm in one network influences the formation of another network in the context of research collaboration. Vonortas and Okamura (2009) consider two types of networks: knowledge networks and partnership networks. The knowledge network is based on patent citations from the past. The authors consider past research joint venture participation from 1985 as the basis of the partnership network. Since research collaboration always involves uncertainty and opportunistic behavior, this literature suggests that previous partner experience and embedded relationships play a key role in overcoming these obstacles. For instance, a firm positioned more centrally has more direct and indirect ties, and this enhances informational dividends in the form of factors like access, timing, and referrals. Therefore, it is anticipated that more central firms are more likely to engage in collaborations than less central ones. Our model, with its complementarity between $\mathbf{H}$ and $\mathbf{G}$, also predicts such an association.

6.3 Community-based Industrialization

In developed countries, with well-defined, transparent legal institutions and easily available public information, the location of industry depends on factors such as geography, resource availability, local demand, and transportation. In the developing world, public channels for information may not exist and communities often provide information. Consequently, economic transactions, including the process of industrialization, occur in the midst of a community and is not necessarily driven by the factors mentioned above. The township-village enterprises of China are probably the best-known examples of this kind. Our model can be used for studying applications in this direction as well. In a community, individuals are connected by pre-existing ties. So, let the community network $\mathbf{H}$ be given and $\mathbf{y}$ represent social activities. When the legal system is not fully developed, and information is available primarily through personal contacts, the network $\mathbf{H}$ will have a significant impact on the formation of $\mathbf{G}$. Clearly, a more socially active player in this scenario can transfer social skills to the sphere of business activity as well.

Banerjee (2000) notes that in socially determined contracts, "*...the community plays a dual role − facilitating contracting between members and perhaps impending contracts between members and outsiders*." Our model suggests that community-based industrialization will deprive an individual who is otherwise productive but socially unimportant. In other words, entrepreneurial excellence can end up being traded off for more connections. Several of the examples mentioned in the introduction regarding credit and ethnicity in Africa already illustrate this. Similar findings have also been indicated by others. Akoten, Swada and Otsuka (2006) while studying the credit access to small and micro industries in Kenya, point out that informal access to credit plays a more significant role in more dense social circles. In a separate study, Nam, Sonobe and Otsuka (2009) document the importance of family ties in the village-based industrialization of Northern Vietnam. Thus social cohesion viewed as a denser inherited network $\mathbf{H}$ can play a significant role in processes like community-based industrialization.

7 Conclusion

In this paper we examine interactions across two networks to determine its impact on endogenous network formation. We assume that players inherit their social network and establish their business links in an economic network with interdepndence between the networks arising from the fact that respective actions in each network are (weak) strategic complements. This is sufficient to show that those who inherit high levels of centrality as a part of an asymmetric social network will be able to form connections in an economic network in which they will have high levels of centrality. We also find that the network formed by the agents has a coarser partition than the inherited network, suggesting the possibility of being able to improve network centrality, but in a limited manner. Thus, our analysis explains why inequality is often entrenched in society, how asymmetries in one network may be magnified or diminished in another, and what determines the identity of players occupying the various vertices of asymmetric equilibrium networks. It thus provides an alternative explanation for the emergence of preferential attachment. This is substantiated by the work of Biggs and Shah (2006) who argue that in Africa ethnic networks raise the performance of insiders (or those with more links) than outsiders (or those with fewer links). Our paper also complements the recent work of Gagnon and Goyal (2017) where players belong to a social network, but can chose to interact either through their network or the market. Their paper focuses on the welfare issues in the context of these institutions. Similar to our findings, they show that when markets and networks are complements, well-connected individuals participate in both institutions which leads to higher welfare and greater inequality, while the opposite happens when the two institutions are substitutes.

We also examine what happens when the fixed network is a KB-regular network, i.e, has a highly symmetric network architecture. In this instance, we find that the endogenously formed network also exhibits symmetry and is either the empty or the complete network. Our paper thus provides a clear picture of the role of path dependence in network formation and how the symmetric or asymmetric architecture of one network influences the equilibrium architecture of another. In doing so, our paper also studies the network formation problem in the context of the growing literature on multiplex networks.

Our paper opens up a number of new directions for future research. The first is an empirical testing of this phenomenon across R&D networks, where firms are a part of multiple collaborative networks along different dimensions of innovative activities, and research efforts in one network are complementary to research efforts in the others. Another important empirical area is in development economics where there is data on individuals being simultaneously involved in multiple relationships like trade, finance, favors, and advice. Of course, there is also the open issue of the impact of one network on another when efforts within networks and across are strategic substitutes (as with public goods), or when efforts display some combination of both strategic complementarity and substitutability within and across networks. Possibly the most important research question relates to multiplex/multigraph network formation. Ever since the seminal papers of Jackson and Wolinsky (1996) and Bala and Goyal (2000), a wide range of theoretical models of network formation have been proposed to address different types of network situations. However, none of these models deal with network formation while taking the interplay between multiple networks into account. We believe that our paper is only a first step and future research will study the simultaneous formation of multiple networks while taking strategic interactions across them into account..

8 Appendix

We begin with a technical result. Let $\mathbf{M}^s = \mathbf{M} \times \mathbf{M} \times \cdots \times \mathbf{M}$ (s times).

Lemma 5 *Suppose* $\mathbf{H}$ *is NSG and let* $\mathbf{M}^s(\mathbf{H},\xi) \equiv \left[m_{ij}^{[s]}\right]$, $s \in \mathbb{Z}_+$. *If* $i \in P_l(\mathbf{H})$ *and* $j \in P_{l'}(\mathbf{H})$ *where* $l' > l$, *then* $m_{ik}^{[s]} < m_{jk}^{[s]}$ $\forall k \in \mathcal{N}$, $\forall s \in \mathbb{Z}_+$. *In particular, for any* $N \times 1$ *vector* $\mathbf{a}$, $\sum_{k=1}^{N} m_{ik}^{[s]} a_k < \sum_{k=1}^{N} m_{jk}^{[s]} a_k$.

Proof: We will first establish that $m_{ik} < m_{jk}$. Consider all walks from i and j respectively to any $k \in \mathcal{N}$. For each walk $ik_1k_2...k_lk$ connecting i to k, there is a corresponding walk of the same length $jk_1k_2...k_lk$ connecting j to k since $k_1 \in \mathbf{N}_i(\mathbf{H}) \subset \mathbf{N}_j(\mathbf{H})$. We now show that there strictly more walks from j to k than from i to k. Suppose $k \in \mathbf{N}_j(\mathbf{H}) \backslash \mathbf{N}_i(\mathbf{H})$. Then there is a walk of length 1 from j to k but no corresponding walk from i to k. Now suppose $k \in \mathbf{N}_i(\mathbf{H})$. There are 3 cases, and in each case j has walks to k for which i does not have corresponding walks to k. (i) $l < l' \leq \lfloor m/2 \rfloor$. There are $d_i(\mathbf{H}) - 1$ walks from i to k and $d_j(\mathbf{H}) - 1$ walks from j to k. Therefore there are $d_j(\mathbf{H}) - d_i(\mathbf{H})$ additional walks of length two from j to k through neighbors in $\mathbf{N}_j(\mathbf{H}) \backslash \mathbf{N}_i(\mathbf{H})$; (ii) $l \leq \lfloor m/2 \rfloor < l'$. In $\mathbf{N}_j(\mathbf{H}) \backslash \mathbf{N}_i(\mathbf{H})$, consider those players whose degree is greater than that of i. These players also have k as a neighbor and therefore j has additional walks of length two to k through these players. Now consider those players in $\mathbf{N}_j(\mathbf{H}) \backslash \mathbf{N}_i(\mathbf{H})$ whose degree is less than that of i. If any of these has k as a neighbor, then once again j has an additional walk of length two to k. If any of these players, say l, does not have k as a neighbor, then j has an additional walk of length three of the form $jljk$. (iii) $l > \lfloor m/2 \rfloor$. The players in $\mathbf{N}_j(\mathbf{H}) \backslash \mathbf{N}_i(\mathbf{H})$ have lower degree than i. For each such player l there is a walk of length three of the form $jljk$, but no corresponding walk from i to k. This establishes $m_{ik} < m_{jk}$. It follows that:

$$m_{ik}^{[2]} = \sum_{l=1}^{N} m_{il} m_{lk} < \sum_{l=1}^{N} m_{jl} m_{lk} = m_{jk}^{[2]}$$

It now follows inductively that $m_{ik}^{[s]} = \sum_{l=1}^{N} m_{il}^{[s-1]} m_{lk} < \sum_{l=1}^{N} m_{jl}^{[s-1]} m_{lk} = m_{jk}^{[s]}$, $\forall s \in \mathbb{Z}_+$. ■

Since the links in the network $\mathbf{H}$ remain fixed and only those in the network $\mathbf{G}$ are allowed to change, it will be convenient to simplify (14) further to reflect this fact. Letting $\gamma \mathbf{M} - \mathbf{I} \equiv [\widehat{m}_{ij}]$, it follows from (11) and (12) that:

$$y_i^* - x_i^* = b_i(\mathbf{H},\psi, \mathbf{1}) + \sum_{j=1}^{N} \widehat{m}_{ij} b_j \left(\mathbf{L}, \boldsymbol{\alpha}_{\mathbf{H}}\right) \equiv \Phi_i(\mathbf{G}, \mathbf{H}) \tag{23}$$

Recall that $\mathbf{b}\left(\mathbf{L}, \boldsymbol{\alpha}_{\mathbf{H}}\right) = \left(\mathbf{I} + \mathbf{L} + \mathbf{L}^2 + \cdots\right) \boldsymbol{\alpha}_{\mathbf{H}}$. Let $\mathbf{L}^s = \left[l_{ij}^{[s]}\right]$ and $\boldsymbol{\alpha}_{\mathbf{H}q} = b_q(\mathbf{H},\psi, \mathbf{1}) + 1$ denote the q^{th} row of the vector $\boldsymbol{\alpha}_{\mathbf{H}}$. Substituting into (23) yields:

$$\Phi_i(\mathbf{G}, \mathbf{H}) = b_i(\mathbf{H},\psi, \mathbf{1}) + \sum_{j=1}^{N} \widehat{m}_{ij} \left(\sum_{s=0}^{\infty} \sum_{q=1}^{N} l_{jq}^{[s]} \boldsymbol{\alpha}_{\mathbf{H}q} \right) \tag{24}$$

Written in this way, only the elements of $[l_{ij}^{[s]}]$ will be influenced by the formation of links in the network $\mathbf{G}$ and we can observe the role that centrality of players in network $\mathbf{H}$ will play in this regard. The incremental utility of player i from adding the link ii' in the network $\mathbf{G}$ is:

$$U_i(\mathbf{G}+ii',\mathbf{H})-U_i(\mathbf{G},\mathbf{H})=\frac{1}{2}\left[\Phi_i(\mathbf{G}+ii',\mathbf{H})+\Phi_i(\mathbf{G},\mathbf{H})\right]\left[\Phi_i(\mathbf{G}+ii',\mathbf{H})-\Phi_i(\mathbf{G},\mathbf{H})\right]+(1-\gamma)\Delta(ii')$$

where $\Delta(ii')\equiv x_i^*(\mathbf{G}+ii',\mathbf{H})y_i^*(\mathbf{G}+ii',\mathbf{H})-x_i^*(\mathbf{G},\mathbf{H})y_i^*(\mathbf{G},\mathbf{H})$. Under the assumption that λ is sufficiently small so that terms with coefficients λ^2 and greater can be ignored, it follows that $\mathbf{L}^s=\lambda s\gamma^{2(s-1)}\mathbf{G}\mathbf{M}^{s-1}+\gamma^{2s}\mathbf{M}^s$. Letting $\mathbf{L}(\mathbf{G}+ii')^s\equiv[(l_{pq}+ii')^{[s]}]$, $\forall s\in\mathbb{Z}_+$,we have:

$$\Phi_i(\mathbf{G}+ii',\mathbf{H})-\Phi_i(\mathbf{G},\mathbf{H})=\left[\sum_{j=1}^{N}\widehat{m}_{ij}\left(\sum_{s=0}^{\infty}\sum_{q=1}^{N}\left\{\left(l_{jq}+ii'\right)^{[s]}-l_{jq}^{[s]}\right\}\boldsymbol{\alpha}_{\mathbf{H}q}\right)\right]$$

Since $\mathbf{M}$ is not affected by link formation in $\mathbf{G}$, we can see that:

$$\mathbf{L}(\mathbf{G}+ii')^s-\mathbf{L}^s=\lambda k\gamma^{2(s-1)}\left[\left(\mathbf{G}+ii'\right)-\mathbf{G}\right]\mathbf{M}^{s-1}$$

All entries in the matrix $[(\mathbf{G}+ii')-\mathbf{G}]$ are zero except for 1's in the (i,i') and $(i'i)$ position. It therefore follows that:

$$\Phi_i(\mathbf{G}+ii',\mathbf{H})-\Phi_i(\mathbf{G},\mathbf{H})=\lambda\sum_{s=1}^{\infty}s\gamma^{2(s-1)}\left\{\widehat{m}_{ii}\sum_{q=1}^{N}m_{i'q}^{[s-1]}\boldsymbol{\alpha}_{\mathbf{H}q}+\widehat{m}_{ii'}\sum_{q=1}^{N}m_{iq}^{[s-1]}\boldsymbol{\alpha}_{\mathbf{H}q}\right\} \tag{25}$$

Also note that player i's Nash equilibrium levels of actions $x_i^*(\mathbf{G},\mathbf{H})$ and $y_i^*(\mathbf{G},\mathbf{H})$ can be written out as:

$$x_i^*(\mathbf{G},\mathbf{H}) = \lambda\sum_{s=1}^{\infty}\left[s\gamma^{2(s-1)}\sum_{q=1}^{N}\sum_{l=1}^{N}g_{il}m_{lq}^{[s-1]}\boldsymbol{\alpha}_{\mathbf{H}q}+\gamma^{2s}\sum_{q=1}^{N}m_{iq}^{[s]}\boldsymbol{\alpha}_{\mathbf{H}q}\right] \tag{26}$$

$$y_i^*(\mathbf{G},\mathbf{H}) = b_i(\mathbf{H},\psi,\mathbf{1})+\gamma\sum_{q=1}^{N}m_{iq}x_q^*(\mathbf{G},\mathbf{H}) \tag{27}$$

We now establish some preliminary results.

Claim 0: *The Nash action levels of a player are strictly increasing in own links.*

Proof: If link $ii'\notin G$, then for action x:

$$x_i^*(\mathbf{G}+ii',\mathbf{H})-x_i^*(\mathbf{G},\mathbf{H}) = \lambda\sum_{s=1}^{\infty}\sum_{q=1}^{N}s\gamma^{2(s-1)}m_{i'q}^{[s-1]}\boldsymbol{\alpha}_{\mathbf{H}q} \tag{28}$$

$$x_j^*(\mathbf{G}+ii',\mathbf{H})-x_j^*(\mathbf{G},\mathbf{H}) = 0,\quad j\neq i,i' \tag{29}$$

Similarly for action level y:

$$y_j^*(\mathbf{G}+ii',\mathbf{H})-y_j^*(\mathbf{G},\mathbf{H})=\gamma m_{ii}\left[x_i^*(\mathbf{G}+ii',\mathbf{H})-x_i^*(\mathbf{G},\mathbf{H})\right] \\ +\gamma m_{ii'}\left[x_{i'}^*(\mathbf{G}+ii',\mathbf{H})-x_{i'}^*(\mathbf{G},\mathbf{H})\right], \quad j\in\mathcal{N} \tag{30}$$

Therefore the Nash actions of player i are strictly increasing in i's links. Also note that the x-action of players not involved in the link remain unaffected while the y-action is strictly increasing. ■

Claim 1: *The increment in Nash action of a player from a link depends only on the identity of the partner.* Formally:

$$\begin{aligned} x_j^*(\mathbf{G}+ij,\mathbf{H})-x_j^*(\mathbf{G},\mathbf{H}) &= x_k^*(\mathbf{G}+ik,\mathbf{H})-x_k^*(\mathbf{G},\mathbf{H}) \\ x_j^*(\mathbf{G}+ij,\mathbf{H})-x_j^*(\mathbf{G},\mathbf{H}) &= x_k^*(\mathbf{G},\mathbf{H})-x_k^*(\mathbf{G}-ik,\mathbf{H}) \end{aligned} \tag{31}$$

Similarly for y_i^*.

Proof: The result for x follows from (28). The result for y follows from (30). ■

Claim 2: *The increment in player i's Nash actions is greater when linking to a partner who is more central in* $\mathbf{H}$. Formally, if $b_j(\mathbf{H},\psi,\mathbf{1})\geq b_k(\mathbf{H},\psi,\mathbf{1})$, then:

$$x_i^*(\mathbf{G}+ij,\mathbf{H})-x_i^*(\mathbf{G},\mathbf{H}) \geq x_i^*(\mathbf{G}+ik,\mathbf{H})-x_i^*(\mathbf{G},\mathbf{H}) \tag{32}$$

$$x_i^*(\mathbf{G}+ij,\mathbf{H})-x_i^*(\mathbf{G},\mathbf{H}) \geq x_i^*(\mathbf{G},\mathbf{H})-x_i^*(\mathbf{G}-ik,\mathbf{H}) \tag{33}$$

Similarly for y_i^*. The inequalities are strict if $b_j(\mathbf{H},\psi,\mathbf{1})>b_k(\mathbf{H},\psi,\mathbf{1})$.

Proof: From Lemma 5, $m_{jq}^{[s]}\geq m_{kq}^{[s]}$, $\forall s\in\mathbb{Z}_+$, and $\forall q\in\mathcal{N}$. The result for x_i^* follows from (28). Since $m_{ij}>m_{ik}$, the result for y_i^* follows from (30). If $b_j(\mathbf{H},\psi,\mathbf{1})>b_k(\mathbf{H},\psi,\mathbf{1})$, then j and k belong to different sets in the degree partition of $\mathbf{H}$ and $m_{jq}^{[s]}>m_{kq}^{[s]}$, $\forall s\in\mathbb{Z}_+$, and $\forall q\in\mathcal{N}$. ■

Claim 3: *Player i's Nash actions are greater than that of player j when i is more central than j in* $\mathbf{H}$ *and the neighborhood of j is contained in that of i.* If $b_i(\mathbf{H},\psi,\mathbf{1})\geq b_j(\mathbf{H},\psi,\mathbf{1})$, and $\mathbf{N}_j(\mathbf{G})\subseteq\mathbf{N}_i(\mathbf{G})$, then:

$$(x_i^*(\mathbf{G},\mathbf{H}),y_i^*(\mathbf{G},\mathbf{H}))\geq\left(x_j^*(\mathbf{G},\mathbf{H}),y_j^*(\mathbf{G},\mathbf{H})\right) \tag{34}$$

The inequalities are strict if $b_i(\mathbf{H},\psi,\mathbf{1})>b_j(\mathbf{H},\psi,\mathbf{1})$ and/or $\mathbf{N}_j(\mathbf{G})\subset\mathbf{N}_i(\mathbf{G})$.

Proof: From Lemma 5, $m_{iq}^{[s]}\geq m_{jq}^{[s]}$, $\forall s\in\mathbb{Z}_+$, $\forall q\in\mathcal{N}$. Since $\mathbf{N}_j(\mathbf{G})\subseteq\mathbf{N}_i(\mathbf{G})$, if $g_{jl}=1$ for some $l\in\mathcal{N}\backslash\{i\}$, then $g_{il}=1$. The result now follows from (26) and (27). The strict inequality case for

$b_i\left(\mathbf{H},\psi,\mathbf{1}\right) > b_j\left(\mathbf{H},\psi,\mathbf{1}\right)$ follows as in the proof of claim 2. For $\mathbf{N}_j(\mathbf{G}) \subset \mathbf{N}_i(\mathbf{G})$, noting that $m_{lq}^{[s-1]} > 0$ $\forall l, q \in \mathcal{N}$, the result follows from (26) by noting that:

$$\sum_{q=1}^{N}\sum_{l=1}^{N} g_{il} m_{lq}^{[s-1]} \boldsymbol{\alpha}_{\mathbf{H}q} > \sum_{q=1}^{N}\sum_{l=1}^{N} g_{jl} m_{lq}^{[s-1]} \boldsymbol{\alpha}_{\mathbf{H}q}$$

This proves the claim. ■

Claim 4: If $b_j\left(\mathbf{H},\psi,\mathbf{1}\right) \geq b_k\left(\mathbf{H},\psi,\mathbf{1}\right)$ and $\mathbf{N}_k(\mathbf{G}) \subseteq \mathbf{N}_j(\mathbf{G})$, then:

$$x_j^*(\mathbf{G}+ij,\mathbf{H}) \geq x_k^*(\mathbf{G}+ik,\mathbf{H}) \tag{35}$$

The inequality is strict if $b_j\left(\mathbf{H},\psi,\mathbf{1}\right) > b_k\left(\mathbf{H},\psi,\mathbf{1}\right)$ and/or $\mathbf{N}_k(\mathbf{G}) \subset \mathbf{N}_j(\mathbf{G})$.

Proof: The result follows from claims 2 and 3. If the inequality for x is strict in (34), then strict inequality holds in (35). ■

Claim 5: If $b_j\left(\mathbf{H},\psi,\mathbf{1}\right) \geq b_k\left(\mathbf{H},\psi,\mathbf{1}\right)$ and $\mathbf{N}_k(\mathbf{G}) \subseteq \mathbf{N}_j(\mathbf{G})$, then $\Delta(ij) \geq \Delta(ik)$. The inequality is strict if $b_j\left(\mathbf{H},\psi,\mathbf{1}\right) > b_k\left(\mathbf{H},\psi,\mathbf{1}\right)$ and/or $\mathbf{N}_k(\mathbf{G}) \subset \mathbf{N}_j(\mathbf{G})$.

Proof: Note that:

$$x_i^*(\mathbf{G},\mathbf{H})y_i^*(\mathbf{G},\mathbf{H}) = b_i\left(\mathbf{H},\psi,\mathbf{1}\right)x_i^*(\mathbf{G},\mathbf{H}) + \gamma x_i^*(\mathbf{G},\mathbf{H})\sum_{q=1}^{N} m_{il}x_i^*(\mathbf{G},\mathbf{H})$$

From claim 1, $x_i^*(\mathbf{G}+ij,\mathbf{H}) \geq x_i^*(\mathbf{G}+ik,\mathbf{H})$. It follows from claims 1-4 that:

$$\begin{aligned}
\Delta(ij) &= b_i\left(\mathbf{H},\psi,\mathbf{1}\right)\left[x_i^*(\mathbf{G}+ij,\mathbf{H}) - x_i^*(\mathbf{G},\mathbf{H})\right] \\
&\quad +\gamma m_{ii}\left[x_i^*(\mathbf{G}+ij,\mathbf{H}) - x_i^*(\mathbf{G},\mathbf{H})\right]\left[x_i^*(\mathbf{G}+ij,\mathbf{H}) + x_i^*(\mathbf{G},\mathbf{H})\right] \\
&\quad +\gamma m_{ij}\left[x_j^*(\mathbf{G}+ij,\mathbf{H}) - x_j^*(\mathbf{G},\mathbf{H})\right]\left[x_j^*(\mathbf{G}+ij,\mathbf{H}) + x_j^*(\mathbf{G},\mathbf{H})\right] \\
&\geq \Delta(ik)
\end{aligned}$$

Strict inequality holds as before. ■

Claim 6: If $b_j\left(\mathbf{H},\psi,\mathbf{1}\right) \geq b_k\left(\mathbf{H},\psi,\mathbf{1}\right)$, then:

$$\Phi_i(\mathbf{G}+ij,\mathbf{H}) - \Phi_i(\mathbf{G},\mathbf{H}) \geq \Phi_i(\mathbf{G}+ik,\mathbf{H}) - \Phi_i(\mathbf{G},\mathbf{H}) \tag{36}$$

$$\Phi_i(\mathbf{G}+ij,\mathbf{H}) - \Phi_i(\mathbf{G},\mathbf{H}) \geq \Phi_i(\mathbf{G},\mathbf{H}) - \Phi_i(\mathbf{G}-ik,\mathbf{H}) \tag{37}$$

$$\Phi_j(\mathbf{G}+ij,\mathbf{H}) - \Phi_j(\mathbf{G},\mathbf{H}) \geq \Phi_k(\mathbf{G},\mathbf{H}) - \Phi_k(\mathbf{G}-ik,\mathbf{H}) \tag{38}$$

The inequality is strict if $b_j\left(\mathbf{H},\psi,\mathbf{1}\right) > b_k\left(\mathbf{H},\psi,\mathbf{1}\right)$.

Proof: The result follows from (25), Lemma 5, and that $\widehat{m}_{jj} \geq \widehat{m}_{kk}$ and $\widehat{m}_{ij} \geq \widehat{m}_{ik}$. Strict inequality holds as before. ■

Proof of Lemma 1: Note that $U_i(\mathbf{G}+ii',\mathbf{H}) - U_i(\mathbf{G},\mathbf{H})$ can be written as:

$$\frac{1}{2}\left[2b_i(\mathbf{H},\psi,\mathbf{1}) + \sum_{q=1}^{N}\widehat{m}_{iq}\left\{x_q^*(\mathbf{G}+ii',\mathbf{H}) + x_q^*(\mathbf{G},\mathbf{H})\right\}\right]\left[\Phi_i(\mathbf{G}+ii',\mathbf{H}) - \Phi_i(\mathbf{G},\mathbf{H})\right]$$
$$+(1-\gamma)\left[x_i^*(\mathbf{G}+ii',\mathbf{H})y_i^*(\mathbf{G}+ii',\mathbf{H}) - x_i^*(\mathbf{G},\mathbf{H})y_i^*(\mathbf{G},\mathbf{H})\right] \tag{39}$$

Then (15) follows by comparing corresponding terms in incremental utility for $U_i(\mathbf{G}+ij,\mathbf{H}) - U_i(\mathbf{G},\mathbf{H})$ and $U_i(\mathbf{G}+ik,\mathbf{H}) - U_i(\mathbf{G},\mathbf{H})$ and using claims 1-6. Inequality (17) follows similarly using $x_q^*(\mathbf{G}+ij,\mathbf{H}) \geq x_q^*(\mathbf{G}-ik,\mathbf{H})$ (from claims 0 and 4), (33), and (37). The strict inequality follows from the fact that $x_i^*(\mathbf{G}+ij,\mathbf{H}) > x_i^*(\mathbf{G}-ik,\mathbf{H})$ from (26). The proof of (16) and (18) follow similarly. ■

Proof of Lemma 2: Given that $\mathbf{G} \subset \mathbf{G}'$, (19) follows from (26), (27), and (39). Existence of a pws-equilibrium follows from Jackson and Watts (2002, Lemma 1). ■

Proof of Lemma 3: By hypothesis, $U_k(\mathbf{G}+ik,\mathbf{H}) - U_k(\mathbf{G},\mathbf{H}) > c$ and $U_i(\mathbf{G}+ik,\mathbf{H}) - U_i(\mathbf{G},\mathbf{H}) > c$. It follows from respectively Lemma 2 and (16) that:

$$U_j(\mathbf{G}'+ij,\mathbf{H}) - U_j(\mathbf{G}',\mathbf{H}) > U_j(\mathbf{G}+ij,\mathbf{H}) - U_j(\mathbf{G},\mathbf{H}) \geq U_k(\mathbf{G}+ik,\mathbf{H}) - U_k(\mathbf{G},\mathbf{H}) > c$$

Player i will reciprocate since Lemma 2 and (15) that:

$$U_i(\mathbf{G}'+ij,\mathbf{H}) - U_i(\mathbf{G}',\mathbf{H}) > U_i(\mathbf{G}+ij,\mathbf{H}) - U_i(\mathbf{G},\mathbf{H}) \geq U_i(\mathbf{G}+ik,\mathbf{H}) - U_i(\mathbf{G},\mathbf{H}) > c$$

This proves the result. ■

Proof of Lemma 4: Let $\mathbf{G}^{(r)}$ denote the interim network at the end of round r (recall that each round has all N players moving in the order of their index). We have already established that $\mathbf{N}_k\left(\mathbf{G}^{(1)}\right) \subseteq \mathbf{N}_j\left(\mathbf{G}^{(1)}\right)$ if $b_j(\mathbf{H},\psi,\mathbf{1}) > b_k(\mathbf{H},\psi,\mathbf{1})$. Suppose this property is true for $r = r'$. We will show that it holds for $r = r'+1$. Let $\mathbf{G}' \supseteq \mathbf{G}^{(r)}$ be the interim network succeeding $\mathbf{G}^{(r)}$ when player k is active in round $r'+1$. Suppose k has a profitable link with player i in $\mathbf{G}'$. Since j is more central in $\mathbf{H}$, j will move after k in round $r'+1$ in some interim network $\mathbf{G}'' \supseteq \mathbf{G}'$. From Lemma 3, j also has a mutually profitable link with i in $\mathbf{G}''$. In addition, in a choice between k and j, any active player in round $r'+1$ realizes greater incremental utility from proposing a link with j due to (15). Therefore $\mathbf{N}_k\left(\mathbf{G}^{(r'+1)}\right) \subseteq \mathbf{N}_j\left(\mathbf{G}^{(r'+1)}\right)$. Therefore the nestedness of neighborhoods requirement in Lemma 3 automatically holds along an improving path and the result follows. ■

Proof of Proposition 2: We have already shown that $P_1(\mathbf{H}) \subseteq P_1(\mathbf{G})$. It only remains to show that for each $i \in P_1(\mathbf{G})$ we have $ij \in \mathbf{G}$ if $j \in P_n(\mathbf{G})$ and $ij \notin \mathbf{G}$ otherwise, i.e. players in $P_1(\mathbf{G})$. There are two cases here: (i) Suppose $P_1(\mathbf{H}) = P_1(\mathbf{G})$. If player i has a mutually profitable link with $j \notin P_n(\mathbf{G})$ in a pws-equilibrium network, then this link was established in some interim network $\mathbf{G}'$ when i was the active player. Since $b_k(\mathbf{H}, \psi, \mathbf{1}) \geq b_i(\mathbf{H}, \psi, \mathbf{1})$ $\forall k \in \mathcal{N} \backslash \{i, j\}$, it follows from Lemma 4 that link jk is mutually profitable in $\mathbf{G}'' \supseteq \mathbf{G}'$. But then player j must be maximally connected in $\mathbf{G}$ which contradicts $j \notin P_n(\mathbf{G})$. (ii) Now suppose $P_1(\mathbf{H}) \subset P_1(\mathbf{G})$. Thus $P_1(\mathbf{G})$ includes players from at least one more partition set in $\mathcal{P}(\mathbf{H})$, say $P_2(\mathbf{H})$. Now suppose $ij \in \mathbf{G}$ for some $i \in P_1(\mathbf{G}) \cap P_2(\mathbf{H})$ and $j \notin P_n(\mathbf{G})$. If ij was established in interim network $\mathbf{G}'$, then from Lemma 4, j has a mutually profitable link with each $i' \in P_1(\mathbf{G}) \cap P_2(\mathbf{H})$ in interim networks $\mathbf{G}'' \supseteq \mathbf{G}'$. In case (i) we have already ruled out a link between $j \notin P_n(\mathbf{G})$ and players in $P_1(\mathbf{H})$. This implies $d_i(\mathbf{G}) > d_k(\mathbf{G})$ for $i \in P_1(\mathbf{G}) \cap P_2(\mathbf{H})$ and $k \in P_1(\mathbf{H})$. But this contradicts $i, k \in P_1(\mathbf{G})$, i.e. the two players should have the same cardinality in $\mathbf{G}$. ■

Proof of Proposition 3: Consider $l = 1$. Suppose that $P_1(\mathbf{H}) \subset P_1(\mathbf{G})$. We will break the argument into the following steps. (i) We show that $P_1(\mathbf{G}) \cap P_2(\mathbf{H}) \neq \emptyset$, i.e. $P_1(\mathbf{G})$ is constructed from successive elements of the partition $\mathcal{P}(\mathbf{H})$. We can prove this by contradiction. Suppose not and assume that $P_1(\mathbf{G})$ "jumps over" elements of $\mathcal{P}(\mathbf{H})$ such that $P_1(\mathbf{G}) \cap P_2(\mathbf{H}) = \emptyset$ but $P_1(\mathbf{G}) \cap P_q(\mathbf{H}) \neq \emptyset$ for some $q > 2$. If $i \in P_1(\mathbf{G}) \cap P_q(\mathbf{H})$ for $q > 2$, and $j \in P_2(\mathbf{G})$, then $b_i(\mathbf{H}, \psi, \mathbf{1}) > b_j(\mathbf{H}, \psi, \mathbf{1})$ implies that every player k who has a mutually profitable link with j will also have a mutually profitable link with i from Lemma 4. This implies $d_i(\mathbf{G}) \geq d_j(\mathbf{G})$ contradicting that $i \in P_1(\mathbf{G})$. (ii) Next we establish the non-partial overlap property that if $P_1(\mathbf{G}) \cap P_2(\mathbf{H}) \neq \emptyset$, then $P_2(\mathbf{H}) \subset P_1(\mathbf{G})$. Suppose that $P_2(\mathbf{H})$ contains a player $r \notin P_1(\mathbf{G})$. Then r is connected to some non-maximally linked player $r' \notin P_n(\mathbf{G})$, since it is players in $P_1(\mathbf{G})$ who are limited to links with the highest degree players in $\mathbf{G}$. But from Lemma 4, player r' will also have a mutually profitable link with $j \in P_1(\mathbf{G}) \cap P_2(\mathbf{H})$. In fact every player who has a link with r also has a mutually profitable link with j. But then $d_j(\mathbf{G}) \geq d_r(\mathbf{G})$ contradicting that $j \in P_1(\mathbf{G})$. It therefore follows that $P_1(\mathbf{G}) = P_1(\mathbf{H}) \cup P_2(\mathbf{H})$ if $P_1(\mathbf{G}) \cap P_2(\mathbf{H}) \neq \emptyset$. (iii) An identical argument establishes the result if $P_1(\mathbf{G}) \cap P_r(\mathbf{H}) \neq \emptyset$ for $r > 2$. An identical argument applies inductively to $l \in \{2, 3, ..., n-1\}$. ■

Proof of Proposition 4: Recall that an improving path leads to $\mathbf{G}^c$ if $0 \leq c < U_1(\mathbf{G}^e + 12, \mathbf{H}) - U_1(\mathbf{G}^e, \mathbf{H})$ and remains at $\mathbf{G}^e$ if $U_{N-1}(\mathbf{G}^e + (N-1, N), \mathbf{H}) - U_{N-1}(\mathbf{G}^e, \mathbf{H}) \leq c$. Consider an intermediate range of costs. The NSG property for $P_1(\mathbf{G})$ and $P_n(\mathbf{G})$ follows from Propositions 1 and 2. Consider the intermediate elements of $\mathcal{P}(\mathbf{G})$ and players $l_1, l_2, .., l_{n-1}$ representing $P_1(\mathbf{G}), P_2(\mathbf{G}), ..., P_{n-1}(\mathbf{G})$ respectively. Consider $l_2 \in P_2(\mathbf{G})$. From the non partial overlap property, $b_{l_1}(\mathbf{H}, \psi, \mathbf{1}) < b_{l_2}(\mathbf{H}, \psi, \mathbf{1})$. Since each player in $P_1(\mathbf{G})$ is directly linked to all players in $P_n(\mathbf{G})$, there must exist a player $k \notin P_n(\mathbf{G})$ such that $kl_2 \in \mathbf{G}$.We now show that $kl_2 \in \mathbf{G}$ for all $k \in P_{n-1}(\mathbf{G}) \cup P_n(\mathbf{G})$. Suppose not and let $kl_2 \in \mathbf{G}$ but $k \notin P_{n-1}(\mathbf{G}) \cup P_n(\mathbf{G})$. Each $l \in P_2(\mathbf{G}) \cup \cdots \cup P_n(\mathbf{G})$ will satisfy $b_l(\mathbf{H}, \psi, \mathbf{1}) \geq b_{l_2}(\mathbf{H}, \psi, \mathbf{1})$. Therefore, from Lemma 4, player k will have mutually profitable links with all players implying that $k \in P_{n-1}(\mathbf{G}) \cup P_n(\mathbf{G})$, a contradiction. Continuing inductively in this manner generates a NSG with the neighborhoods of low degree players nested in the neighborhoods of high degree players. Finally, note that there can be at most one non-singleton component. If to the contrary there exists at least two non-singleton components in a pws-equilibrium graph, then we can identify players i, j, k, l such that $ij \in \mathbf{G}$, $kl \in \mathbf{G}$, and $b_i(\mathbf{H}, \psi, \mathbf{1}) \leq b_k(\mathbf{H}, \psi, \mathbf{1}) \leq b_j(\mathbf{H}, \psi, \mathbf{1}) \leq b_l(\mathbf{H}, \psi, \mathbf{1})$. But then players j and l can mutually benefit from a link. ■

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